Asymptotics of some Solutions to the Cylindrical Toda Equations

Cylindrical Toda equations:

$$q_k''(t)+\frac{1}{t}\,q_k'(t)=4\left(e^{q_k-q_{k-1}}-e^{q_{k+1}-q_k}\right)\quad (k\in\mathbf{Z}).$$

We determine asymptotics of some solutions given in terms of operator determinants.

Define kernels $K_k(u,v)$ on $L^2(\mathbf{R}^+)$ by

$$\int\frac{e^{-t[(1-\omega)u+(1-\omega^{-1})u^{-1}]}}{-\omega u+v}\,\omega^k\,d\rho(\omega),$$

where $\mathrm{Re}\,\omega<1,\ \mathrm{Re}\,\omega^{-1}<1$ for $\omega\in\mathrm{supp}\,\rho$.

Widom (1997):

$$q_k(t)=\log\det(I-K_k)-\log\det(I-K_{k-1})$$

solves the cylindrical Toda equations.

If ρ supported in nth roots of unity then the solution is n-periodic.

If n=2 then $\operatorname{supp}\rho = \{-1\}$. If $\rho(\{-1\}) = \lambda$ then

$$K_0(u,v) = \lambda\,\frac{e^{-2t\,(u+u^{-1})}}{u+v}.$$

Equation for q_0 becomes cylindrical sinh-Gordon equation

$$q''(t) + t^{-1}\,q(t) = 8\,\sinh 2q.$$

(McCoy-Tracy-Wu 1977)

If $n = 3$ and $\rho(\{e^{2\pi i/3}\}) + \rho(\{e^{-2\pi i/3}\}) = 0$ equation for q_0 becomes cylindrical Bullough-Dodd equation

$$q''(t) + t^{-1}\,q(t) = 4\,e^{2q} - 4\,e^{-q}.$$

Problem: Determine asymptotics as $t \to 0$.

Enough to consider $\det(I - K_0)$. Define

$$h(s) = 1 - \frac{\pi}{\sin \pi s}\int (-\omega)^{s-1}\,d\rho(\omega).$$

After variable change $s = \frac{1}{2} - i\xi$ this is symbol of an associated Wiener-Hopf operator.

Regular case:

$$h(s) \neq 0 \text{ for } \operatorname{Re} s = \frac{1}{2}, \quad \arg h(s)\Big|_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} = 0.$$

Singular cases: Everything else.

McCoy-Tracy-Wu kernel:

$$h(s) = 1 - \lambda\,\frac{\pi}{\sin \pi s}.$$

Regular if $\lambda \notin [\pi^{-1}, \infty)$.

Tracy-Widom (1998). In the regular case, as $t \to 0$

$$\det\,(I - K_0) \sim b\,\left(\frac{t}{n}\right)^a$$

where a, b have integral representations.

In the n-periodic case $h(s)$ has n zeros α_k in the strip $\frac{1}{2} < \operatorname{Re} s < n + \frac{1}{2}$.

$$a = \frac{1}{n} \sum \alpha_k^2 - \frac{(n+1)(2n+1)}{6},$$

$$b = \frac{\prod_{|j|<n} G(\frac{j}{n}+1)^{n-|j|}}{\prod_{j,k} G(\frac{\alpha_j-\alpha_k}{n}+1)}.$$

Here G denotes the Barnes G-function.

The T-W method involved resolvent kernel. We describe another approach which gives a more general result and extends to some singular cases.

Make variable changes so operator acts on $L^2(\mathbf{R})$, set $t = e^{-\alpha}$. Kernel becomes

$$K_\alpha(x,y) = \int \frac{e^{-[(1-\omega)e^{x-\alpha}+(1-\omega^{-1})e^{-x-\alpha}]}}{-\omega\, e^{(x-y)/2} + e^{(y-x)/2}} d\rho(\omega).$$

Want asymptotics of $\det(I - K_\alpha)$ as $\alpha \to +\infty$.

There is a limiting kernel

$$K(x,y) = \int \frac{1}{-\omega\, e^{(x-y)/2} + e^{(y-x)/2}}\, d\rho(\omega) = k(x-y).$$

Kernel of $I - K$ is $\delta(x-y) - k(x-y)$. Symbol is Fourier transform of $\delta - k$,

$$\sigma(\xi) = (\widehat{\delta - k})\,(\xi) = h(\tfrac{1}{2} - i\xi).$$

The truncated Wiener-Hopf operator $W_{(-\alpha,\,\alpha)}(\sigma)$ is $I - K$ acting on $L^2(-\alpha,\, \alpha)$.

Kac-Achieser: If

$$\sigma(\xi) \neq 0 \text{ for } \xi \in \mathbf{R}, \quad \arg \sigma(\xi)|_{-\infty}^{\infty} = \mathbf{0}$$

(plus technical conditions) then

$$\det W_{(-\alpha,\,\alpha)}(\sigma) \sim G(\sigma)^{2\alpha}\, E(\sigma),$$

with explicit $G(\sigma)$ and $E(\sigma)$.

This suggests using Kac-Achieser for our problem of $\det(I - K_\alpha)$.

Recall

$$K_\alpha(x,y) = \int \frac{e^{-[(1-\omega)e^{x-\alpha}+(1-\omega^{-1})e^{-x-\alpha}]}}{-\omega\, e^{(x-y)/2} + e^{(y-x)/2}} d\rho(\omega).$$

Observe $e^{-x-\alpha} \to 0$ as $\alpha \to \infty$ uniformly for $x > 0$, and $K_\alpha(x,\, y)$ becomes translate of

$$K_+(x,\, y) = \int \frac{e^{-(1-\omega)e^{x}}}{-\omega\, e^{(x-y)/2} + e^{(y-x)/2}} d\rho(\omega).$$

When $x < 0$ limiting kernel becomes translate of

$$K_-(x,\, y) = \int \frac{e^{-(1-\omega)e^{-x}}}{-\omega\, e^{(x-y)/2} + e^{(y-x)/2}} d\rho(\omega).$$

General setting: Family of trace class operators K_α on $L^2(\mathbf{R})$ with three associated operators K, $K_\pm$ where $K(x,y) = k(x-y)$.

Notation: $\chi^\pm$ multiplication by $\chi_{\mathbf{R}^\pm}(x)$; $(T_\alpha f)(x) = f(x-\alpha)$; $o_1(1)$ family of operators whose trace norms are $o(1)$.

Assumptions:

$$\chi^-(K_\alpha - K)\chi^+ = o_1(1),\ \ \chi^+(K_\alpha - K)\chi^- = o_1(1).$$

$$\chi^+(K_\alpha - T_\alpha K_+ T_{-\alpha})\chi^+ = o_1(1),$$

$$\chi^-(K_\alpha - T_{-\alpha} K_- T_\alpha)\chi^- = o_1(1),$$

$L_- := K_- - \chi^+ K \chi^+$, $L_+ := K_+ - \chi^- K \chi^-$ tr. class.

In the regular case the Wiener-Hopf operators $W^\pm(\sigma) = I - \chi^\pm K \chi^\pm$ are invertible.

Theorem:

$$\det(I - K_\alpha) \sim G(\sigma)^{2\alpha} E(\sigma)$$

$$\times \det(I - W^+(\sigma)^{-1} L_-)\ \det(I - W^-(\sigma)^{-1} L_+).$$

Outline of derivation. Think of $L^2(\mathbf{R})$ as $L^2(\mathbf{R}^-)\oplus L^2(\mathbf{R}^+)$. Then K_α has matrix representation

$$\begin{pmatrix} \chi^- K_\alpha \chi^- & \chi^- K_\alpha \chi^+ \\ \chi^+ K_\alpha \chi^- & \chi^+ K_\alpha \chi^+ \end{pmatrix}.$$

With error $o_1(1)$ this equals

$$\begin{pmatrix} \chi_{(-\alpha,0)} K \chi_{(-\alpha,0)} & \chi_{(-\alpha,0)} K \chi_{(0,\alpha)} \\ \chi_{(0,\alpha)} K \chi_{(-\alpha,0)} & \chi_{(0,\alpha)} K \chi_{(0,\alpha)} \end{pmatrix}$$

$$+\begin{pmatrix} \chi^- T_{-\alpha} L_- T_\alpha \chi^- & 0 \\ 0 & \chi^+ T_\alpha L_+ T_{-\alpha} \chi^+ \end{pmatrix}.$$

I minus the first operator is $I-\chi_{(-\alpha,\alpha)} K \chi_{(-\alpha,\alpha)} = W_{(-\alpha,\alpha)}(\sigma)$. Factor it out on the left. Then think of operators as acting on $L^2(\mathbf{R})\oplus L^2(\mathbf{R})$.

Pull the translation operators out and then multiply by $\begin{pmatrix} T_{-\alpha} & 0 \\ 0 & T_\alpha \end{pmatrix}$ on the right and $\begin{pmatrix} T_\alpha & 0 \\ 0 & T_{-\alpha} \end{pmatrix}$ on the left.

Upper-left corner becomes

$\chi_{(-\infty,\alpha)}\,(I-\chi_{(0,2\alpha)}\,K\,\chi_{(0,2\alpha)})^{-1}\,\chi_{(-\infty,\alpha)}\,L_-\,\chi_{(-\infty,\alpha)}.$
Use fact that $(I-\chi_{(0,2\alpha)}\,K\,\chi_{(0,2\alpha)})^{-1}$ converges strongly to $W^+(\sigma)^{-1}$. Lower-right corner similar; off-diagonal terms are $o_1(1)$.

Therefore operator matrix converges in trace norm to

$$\begin{pmatrix} I - W^+(\sigma)^{-1}\,L_- & 0 \\ 0 & I - W^-(\sigma)^{-1}\,L_+ \end{pmatrix},$$

whose determinant is

$$\det\,(I - W^+(\sigma)^{-1}\,L_-)\ \det\,(I - W^-(\sigma)^{-1}\,L_+).$$

How to evaluate this for Toda kernels?

Use general fact

$$\log\,\det\,(I-T) = -\int_0^1 \mathrm{tr}\,T^{-1}\,(I-\lambda\,T)^{-1}\,d\lambda.$$

Want to compute $(I-W^+(\sigma)^{-1}\,L_-)^{-1}$. Factor out $W^+(\sigma)=I-\chi^+\,K\,\chi^+$, left with

$$W^+(\sigma)-L_-=I-K_-.$$

There is a representation $K_-=AB$ where $BA=K$ acting on $L^2(\mathbf{R}^+)$. Wiener-Hopf operator, so we know $(I-BA)^{-1}$. Then use

$$(I-AB)^{-1}=I+A\,(I-BA)^{-1}\,B.$$

What about singular cases?

Used regularity for Kac-Achieser and strong convergence of $W_{(0,\,\alpha)}^{-1}$. In singular cases these don't hold and need replacing. Can do this for some singular cases when σ at edge of regularity (simplest example $\xi^2/(1+\xi^2)$). Get asymptotics

$$\det\,(I-K_\alpha)\sim\alpha\,G(\sigma)^{2\alpha}\,F(\sigma)$$

$$\times\det(I-W^+(\sigma)^{-1}\,L_-)\;\det(I-W^-(\sigma)^{-1}\,L_+)$$

with $F(\sigma)$ given by an integral formula, $W^\pm(\sigma)^{-1}$ act between weighted L^2 spaces.

$\sim$ can be replaced by asymptotic and convergent series $1+p_1\,\alpha^{-1}+p_2\,\alpha^{-2}+\cdots$.

In literature asymptotics of $q = q_0$ in the following cases:

McCoy-Tracy-Wu when $\rho(\{-1\}) = \lambda = \pi^{-1}$. K_0 singular, K_{-1} regular.

$$e^{q(t)} = 2\,t\,(\,\log t^{-1} + \log 2 - \gamma + O(1/\log t)).$$

Bullough-Dodd, case $n = 3$ if $\rho(\{e^{2\pi i/3}\}) + \rho(\{e^{-2\pi i/3}\}) = 0$.

If $\lambda = 2\pi i\,\rho(\{e^{-2\pi i/3}\})$ then

$$h(s) = 1 + \lambda \frac{\sin \pi((s+2)/3)}{\sin \pi s}.$$

When $\lambda = 1$ K_0 regular, K_{-1} singular. Our result applies.

$$e^{-q(t)} = 2\,t\,(\,\log t^{-1} + \log 3 - \frac{2}{3}\log 2 - \gamma + O(1/\log t)).$$

Obtained by Kitaev (1989).

When $\lambda = -3$ K_{-1} regular, K_0 singular. Our result does not apply. Kitaev derives

$$e^{q(t)} \sim 2\,t^2\,(\log t)^2.$$