

Limiting Laws of Linear Eigenvalue Statistics for Unitary Invariant Matrix Models

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Abstract

We study the variance and the Laplace transform of the probability law of linear eigenvalue statistics of unitary invariant Matrix Models of $n \times n$ Hermitian matrices as $n \rightarrow \infty$. Assuming that the test function of statistics is smooth enough and using the asymptotic formulas by Deift et al for orthogonal polynomials with varying weights, we show first that if the support of the Density of States of the model consists of $q \geq 2$ intervals, then in the global regime the variance of statistics is a quasiperiodic function of n as $n \rightarrow \infty$ generically in the potential, determining the model. We show next that the exponent of the Laplace transform of the probability law is not in general $1/2 \times$ variance, as it should be if the Central Limit Theorem would be valid, and we find the asymptotic form of the Laplace transform of the probability law in certain cases.

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1 Introduction

Random matrix theory deals mostly with eigenvalue distributions of various ensembles of $n \times n$ random matrices as $n \rightarrow \infty$. Many questions of this branch of the theory can be formulated in terms of the Eigenvalue Counting Measure \mathcal{N}_n , $\mathcal{N}_n(\Delta_n)$ is the number of eigenvalues in a given part Δ_n of the spectrum of a matrix M_n . In a particular case of $n \times n$ Hermitian matrices eigenvalues are real and we can write for $\Delta_n \subset \mathbb{R}$:

$$\mathcal{N}_n(\Delta_n) := \#\{\lambda_l^{(n)} \in \Delta_n, l = 1, \dots, n\} = \sum_{l=1}^n \chi_{\Delta_n}(\lambda_l^{(n)}), \quad (1.1)$$

where χ_{Δ} is the indicator of Δ . A more general object is a linear eigenvalue statistic, defined as

$$\mathcal{N}_n[\varphi_n] := \sum_{l=1}^n \varphi_n(\lambda_l^{(n)}) = \text{Tr } \varphi_n(M_n) = \int_{\mathbb{R}} \varphi_n(\lambda) \mathcal{N}_n(d\lambda) \quad (1.2)$$

for a certain test function

$$\varphi_n : \mathbb{R} \rightarrow \mathbb{C} \quad (1.3)$$

It is known that in many cases there exists a scaling of matrix entries (i.e. a choice of the scale of the spectral axis), such that for a sufficiently big class of n -independent intervals in (1.1) (test functions in (1.2)) the Normalized Counting Measure of eigenvalues

$$N_n = \mathcal{N}_n / n \quad (1.4)$$

converges weakly to a non-random measure N , known as the Integrated Density of States measure (IDS) of the ensemble

$$N_n \rightarrow N. \quad (1.5)$$

The corresponding scale (asymptotic regime) is called the *global* (or *macroscopic*). The convergence is either in probability or even with probability 1. We refer the reader to works [19, 9, 30], where this fact is proved and discussed for two most widely studied classes of random matrix ensembles, the Wigner Ensembles (independent or weakly dependent entries) and the Matrix Models (invariant matrix probability laws). Analogous facts are also known for many other ensembles.

Besides, in many cases the measure N possesses a bounded and continuous density [16, 29]:

$$N(d\lambda) = \rho(\lambda)d\lambda, \quad (1.6)$$

We call ρ the Density of States (DOS) of ensemble.

These results can be viewed as analogs of the Law of Large Numbers of probability theory. Hence, a natural next question, important also for applications, concerns the limiting probability law of fluctuation of the Normalized Counting Measures of eigenvalues or their linear statistics, i.e. an analog of the Central Limit Theorem of the probability theory. The question is not going to be completely trivial, because eigenvalues of random matrices are strongly dependent, and, as a result, the variance of the linear statistics (1.2) with a C^1 test function does not grow with n (see e.g. formulas (2.36) – (2.37) below). Nevertheless, it was found that in a variety of cases fluctuations of various spectral characteristics of eigenvalues of random matrix ensembles are asymptotically Gaussian (see e.g. [5, 6, 13, 15, 18, 19, 21, 23, 24, 25, 35, 36, 37, 38, 39]). In particular, for the global scale and a C^1 test function

in (1.2) this requires, roughly speaking, the same order of magnitude of the entries for the Wigner Ensembles (see [19, 25] for exact conditions, similar to the Lindeberg condition of the probability theory), and for the Matrix Models one needs to assume that the support of the IDS (1.5) is a connected interval of the spectral axis [23]: $\text{supp } N = [a, b]$. The last result was obtained by using variational techniques, introduced in the random matrix theory in paper [9] in order to prove (1.5).

Being applicable to Matrix Models of all three symmetry classes of the random matrix theory (real symmetric, Hermitian and quaternion real matrices), the variational techniques were efficient so far in the study of fluctuations of eigenvalue statistics only in the case where $\text{supp } N = [a, b]$. In this paper we consider only the Matrix Models of Hermitian matrices, but for a general case of a multi-interval support of N :

$$\sigma := \text{supp } N = \bigcup_{l=1}^q [a_l, b_l], \quad q \geq 1. \quad (1.7)$$

In this case we can use recent powerful results by Deift et al [17] on asymptotics of special class of orthogonal polynomials. We find that if φ is real analytic, then the traditional Central Limit Theorem is not always valid for the case $q \geq 2$. In particular, the variance and the probability law oscillate in n as $n \rightarrow \infty$, hence their limiting form depend on a sequence $n_j \rightarrow \infty$. Moreover, the limiting probability laws are not always Gaussian.

The paper is organized as follows. In Section 2 we study the variance of linear eigenvalue statistics in the global regime, i.e., for n -independent φ in (1.2), confining ourselves mostly to the case of C^1 test functions φ . We find that the variance is quasi-periodic in n in general and its frequency module is determined by the charges

$$\beta_l = N([a_{l+1}, \infty)), \quad l = 1, \dots, q-1, \quad (1.8)$$

determined by the IDS of (1.5) and its support (1.7). Hence, the variance has no limit as $n \rightarrow \infty$ for $q \geq 2$ and its asymptotic forms are indexed by points of the subset $\mathbb{H}^{q-1} \subset \mathbb{T}^{q-1}$, which is the closure of limit points of the vectors

$$(\{\beta_1 n\}, \dots, \{\beta_{q-1} n\}) \in \mathbb{T}^{q-1}, \quad (1.9)$$

where $\{\beta_l n\}$, $l = 1, \dots, q-1$ are the fractional parts of $\beta_l n$, $l = 1, \dots, q-1$. This phenomenon has been already found in certain cases [3, 8], but we give its general description.

In Section 3 we study the Laplace transform of the probability law of linear eigenvalue statistics (1.2) in the global regime, passing to the limit along a subsequence

$$(\{\beta_1 n_j(x)\}, \dots, \{\beta_{q-1} n_j(x)\}) \rightarrow x \in \mathbb{H}^{q-1}, \quad (1.10)$$

and confining ourselves to real analytic test functions. We give first a general formula for the corresponding limit. Since the formula is rather complex, we consider several particular cases, where we show that the exponent of the limiting (in the sense (1.10)) Laplace transform is not quadratic in φ , hence the limiting law is not Gaussian (see formulas (3.1), (3.13), and (3.19) – (3.20)). This has to be compared with results of paper [14], according to which the limits of variance and the probability law are the same for all sequences $n_j \rightarrow \infty$ (i.e., exist), and the limiting probability law is Gaussian.

The random matrix theory deals also with two more asymptotic regimes in addition to the global one. Namely, if, having fixed the global scale allowing us to prove (1.5), we set in (1.2)

$$\varphi_n(\lambda) = \varphi((\lambda - \lambda_0)n^\alpha), \quad 0 < \alpha < 1, \quad \lambda_0 \in \text{supp } N, \quad (1.11)$$

where φ is n -independent, we obtain the *intermediate* regime, and if

$$\varphi_n(\lambda) = \varphi((\lambda - \lambda_0)n), \quad \lambda_0 \in \text{supp } N \quad (1.12)$$

then we have the *local* (or *microscopic*) regime. In Section 4 we discuss the form of variance and the validity of the CLT in these regimes.

In Appendix A.1 we find the asymptotic form of the variance of the Eigenvalue Counting Measure (1.1) (corresponding to a piece-wise constant $\varphi_n = \chi_\Delta$ in (1.2)) for the Gaussian Unitary Ensemble (GUE). The variance proves to be $O(\log n)$ instead of being $O(1)$ in the case of a C^1 test function. This shows that the Lipschitz condition (2.35) on test functions that we use to prove that the variance is bounded in n is pertinent (for a more precise condition see [18]). In Appendix A.2 we compute the variational derivative of β_1 in the case $q = 2$, which is used in Section 4, and discuss related topics.

We note that a completely rigorous derivation of the results of this paper, especially those of Section 3, requires rather technical and tedious arguments. They will not be presented below. Rather, we confine ourselves to the presentation of results, their discussion, and outline of corresponding proofs.

2 Variance of linear eigenvalue statistics

2.1 Generalities

Recall that unitary invariant Matrix Models are $n \times n$ Hermitian random matrices, defined by the probability law

$$\mathbf{P}_n(dM_n) = Z_n^{-1} \exp\{-n\text{Tr}V(M_n)\}dM_n, \quad (2.1)$$

where $M_n = \{M_{jk}\}_{j,k=1}^n$, $M_{jk} = \overline{M_{kj}}$,

$$dM_n = \prod_{j=1}^n dM_{jj} \prod_{1 \leq j < k \leq n} d\Re M_{jk} d\Im M_{jk},$$

and $V : \mathbb{R} \rightarrow \mathbb{R}_+$ is a measurable function, called the potential and such that

$$V : \mathbb{R} \rightarrow \mathbb{R}_+, \quad V(\lambda) \geq (2 + \delta) \log |\lambda|, \quad |\lambda| > L, \quad (2.2)$$

for some positive δ and L .

The limit (1.5) can be described as follows [9, 23]. Consider the functional:

$$\mathcal{E}_V[m] = - \int_{\mathbb{R}} \int_{\mathbb{R}} \log |\lambda - \mu| m(d\lambda) m(d\mu) + \int_{\mathbb{R}} V(\lambda) m(d\lambda), \quad (2.3)$$

where m is a non-negative unit measure.

The variational problem, defined by (2.3), goes back to Gauss and is called the minimum energy problem in the external field V . The unit measure N minimizing (2.3) is called the

equilibrium measure in the external field V because of its evident electrostatic interpretation as the equilibrium distribution of linear charges on the ideal conductor occupying the axis \mathbb{R} and confined by the external electric field of potential V . We stress that the respective minimizing procedure determines both the support σ of the measure and the form of the measure. This should be compared with the variational problem of the theory of logarithmic potential, where the external field is absent but the support σ is given (see (2.50)). The minimum energy problem in the external field (2.3) arises in various domains of analysis and its applications (see recent book [34] for a rather complete account of results and references concerning the problem).

The measure N and its support σ are uniquely determined by the Euler-Lagrange equation of the variational problem [9, 34]:

$$V_{eff}(\lambda) = F, \lambda \in \sigma, \quad (2.4)$$

$$V_{eff}(\lambda) \geq F, \lambda \notin \sigma, \quad (2.5)$$

where

$$V_{eff}(\lambda) = V(\lambda) - 2 \int_{\sigma} \log |\lambda - \mu| N(d\mu), \quad (2.6)$$

and F is a constant (the Lagrange multiplier of the normalization condition $N(\mathbb{R}) = 1$).

According to [9] (see also [23]), if the potential V in (2.1) – (2.2) satisfies the local Lipschitz condition

$$|V(\lambda_1) - V(\lambda_2)| \leq C|\lambda_1 - \lambda_2|^\gamma, \quad |\lambda_1|, |\lambda_2| \leq L, \quad (2.7)$$

valid for any $L > 0$ and some positive C and γ , then (1.5) holds with probability 1, and N is the minimizer of (2.3). Moreover, if V' satisfies the local Lipschitz condition, and the support (1.7) is a finite union of disjoint finite intervals, then (1.6) is valid [16, 29] and the Density of States can be written as

$$\rho(\lambda) = P(\lambda) \sqrt{R_q(\lambda)}, \quad \lambda \in \sigma, \quad (2.8)$$

where $P(\lambda)$ is a continuous function,

$$\sqrt{R_q(\lambda)} = \sqrt{R_q(z)} \Big|_{z=\lambda+i0}, \quad R_q(z) = \prod_{l=1}^q (z - a_l)(z - b_l), \quad (2.9)$$

and $\sqrt{R_q(z)}$ is the branch, determined by the condition: $\sqrt{R_q(z)} = z^q + O(z^{q-1})$, $z \rightarrow \infty$. To obtain these formulas, provided that the support (1.7) is given, we differentiate (2.4), (2.6) and obtain the singular integral equation

$$\text{v.p.} \int_{\sigma} \frac{\rho(\mu) d\mu}{\mu - \lambda} = -\frac{V'(\lambda)}{2}, \quad \lambda \in \sigma. \quad (2.10)$$

Then the bounded solution of the equation has the form (2.8) (see e.g. [28]) in which

$$P(\lambda) = \frac{1}{2\pi^2} \int_{\sigma} \frac{V'(\mu) - V'(\lambda)}{\mu - \lambda} \frac{d\mu}{\sqrt{R_q(\mu)}}. \quad (2.11)$$

The endpoints of the support are rather complex functionals of the potential in general. Thus, it is of interest to mention a simple case [11].

Let $v : \mathbb{R} \rightarrow \mathbb{R}$ be a monic polynomial of degree q with real coefficients. Assume that for some $g > 0$ all zeros of $v^2 - 4g$ are real and simple and set

$$V(\lambda) = \frac{v^2(\lambda)}{2gq}. \quad (2.12)$$

Then the DOS of the matrix model (2.1) with this potential is

$$\rho(\lambda) = \frac{|v'(\lambda)|}{2\pi gq} |v^2(\lambda) - 4g|^{1/2}, \quad \lambda \in \sigma, \quad (2.13)$$

where

$$\sigma = \{\lambda \in \mathbb{R} : v^2(\lambda) \leq 4g\}. \quad (2.14)$$

Besides, in this case we have for the charges (1.8)

$$\beta_l = \frac{q-l}{q}, \quad l = 1, \dots, q-1, \quad (2.15)$$

hence the set \mathbb{H}^{q-1} is $\{0, 1/q, \dots, (q-1)/q\}$.

The case $q = 1$ corresponds to the Gaussian Unitary Ensemble and (2.13) yields the semi-circle law by Wigner:

$$V = \frac{\lambda^2}{2g}, \quad \sigma = [-2g, 2g], \quad \rho(\lambda) = \frac{1}{2\pi g} \begin{cases} \sqrt{4g - \lambda^2}, & \lambda \in \sigma, \\ 0, & \lambda \notin \sigma. \end{cases} \quad (2.16)$$

In the case $q = 2$ and

$$v(\lambda) = \lambda^2 - m^2, \quad m^2 > 2\sqrt{g}, \quad (2.17)$$

we have:

$$\sigma = [-b, -a] \cup [a, b], \quad a = \sqrt{m^2 - 2\sqrt{g}}, \quad b = \sqrt{m^2 + 2\sqrt{g}}, \quad (2.18)$$

and

$$\rho(\lambda) = \frac{|\lambda|}{2\pi g} \begin{cases} \sqrt{(b^2 - \lambda^2)(\lambda^2 - a^2)}, & \lambda \in \sigma, \\ 0, & \lambda \notin \sigma. \end{cases} \quad (2.19)$$

We will use in this paper the expressions for the variance of linear statistics (1.2) and for the Laplace transform of their probability law via special orthogonal polynomials. The technique dates back to works by Dyson, Gaudin, Mehta, and Wigner of the 60s (see e.g. [27]). Namely, we have for the joint probability density of eigenvalues of ensemble (2.1):

$$p_n(\lambda_1, \dots, \lambda_n) = \left(\det \{ \psi_{j-1}^{(n)}(\lambda_k) \}_{j,k=1}^n \right)^2 / n!, \quad (2.20)$$

where

$$\psi_l^{(n)} = e^{-nV/2} P_l^{(n)}, \quad (2.21)$$

and

$$\{P_l^{(n)}\}_{l \geq 0} \quad (2.22)$$

is the system of orthonormal polynomials with respect to the weight

$$w_n = e^{-nV}, \quad (2.23)$$

so that

$$\int_{\mathbb{R}} e^{-nV(\lambda)} P_l^{(n)}(\lambda) P_m^{(n)}(\lambda) d\lambda = \delta_{l,m}, \quad l, m = 0, 1, \dots \quad (2.24)$$

The polynomials satisfy the three - term recurrence relation for $l = 0, 1, \dots$:

$$r_l^{(n)} \psi_{l+1}^{(n)}(\lambda) + s_l^{(n)} \psi_l^{(n)}(\lambda) + r_{l-1}^{(n)} \psi_{l-1}^{(n)}(\lambda) = \lambda \psi_l^{(n)}(\lambda), \quad r_{-1}^{(n)} = 0, \quad (2.25)$$

thereby determining a semi - infinite Jacobi matrix:

$$J_{j,k}^{(n)} = r_j^{(n)} \delta_{j+1,k} + s_k^{(n)} \delta_{j,k} + r_{j-1}^{(n)} \delta_{j-1,k}, \quad j, k = 0, 1, \dots \quad (2.26)$$

By using (2.20) it can be shown that [33]

$$\mathbf{Var}\{\mathcal{N}_n[\varphi]\} = \frac{1}{2} \int_{\mathbb{R}} \int_{\mathbb{R}} |\varphi(\lambda_1) - \varphi(\lambda_2)|^2 K_n^2(\lambda_1, \lambda_2) d\lambda_1 d\lambda_2 \quad (2.27)$$

$$= \int_{\mathbb{R}} \int_{\mathbb{R}} \left| \frac{\Delta\varphi}{\Delta\lambda} \right|^2 \mathcal{V}_n(\lambda_1, \lambda_2) d\lambda_1 d\lambda_2, \quad (2.28)$$

where

$$K_n(\lambda_1, \lambda_2) = \sum_{l=0}^{n-1} \psi_l^{(n)}(\lambda_1) \psi_l^{(n)}(\lambda_2) \quad (2.29)$$

is known as the reproducing kernel of the system (2.21),

$$\frac{\Delta\varphi}{\Delta\lambda} = \frac{\varphi(\lambda_1) - \varphi(\lambda_2)}{\lambda_1 - \lambda_2}, \quad (2.30)$$

and

$$\mathcal{V}_n(\lambda_1, \lambda_2) = \left[r_{n-1}^{(n)} \left(\psi_n^{(n)}(\lambda_1) \psi_{n-1}^{(n)}(\lambda_2) - \psi_{n-1}^{(n)}(\lambda_1) \psi_n^{(n)}(\lambda_2) \right) \right]^2 / 2. \quad (2.31)$$

Note that in passing from (2.27) to (2.28) we used the Christoffel-Darboux formula [40]

$$K_n(\lambda_1, \lambda_2) = r_{n-1}^{(n)} \frac{\psi_n^{(n)}(\lambda_1) \psi_{n-1}^{(n)}(\lambda_2) - \psi_{n-1}^{(n)}(\lambda_1) \psi_n^{(n)}(\lambda_2)}{\lambda_1 - \lambda_2}. \quad (2.32)$$

It is of interest to have a spectral-theoretic interpretation of the above formulas. Notice that $J^{(n)}$ of (2.26) determines a selfadjoint operator, that we denote again $J^{(n)}$. It acts in $l^2(\mathbb{Z}_+)$, and the matrix

$$\mathcal{E}^{(n)}(d\lambda) = e^{(n)}(\lambda) d\lambda, \quad e^{(n)}(\lambda) = \{e_{lm}^{(n)}(\lambda)\}_{l,m=0}^{\infty}, \quad e_{lm}^{(n)}(\lambda) = \psi_l^{(n)}(\lambda) \psi_m^{(n)}(\lambda) \quad (2.33)$$

is its resolution of identity [1]. This allows us to write (2.31) in the form

$$\begin{aligned} \mathcal{V}_n(\lambda_1, \lambda_2) &= (r_{n-1}^{(n)})^2 (e_{n,n}^{(n)}(\lambda_1) e_{n-1,n-1}^{(n)}(\lambda_2) \\ &+ e_{n,n}^{(n)}(\lambda_2) e_{n-1,n-1}^{(n)}(\lambda_1) - 2e_{n-1,n}^{(n)}(\lambda_1) e_{n,n-1}^{(n)}(\lambda_2)) / 2. \end{aligned} \quad (2.34)$$

Assume now that φ is of the class C^1 and

$$\sup_{\lambda \in \mathbb{R}} |\varphi'(\lambda)| < \infty. \quad (2.35)$$

for some positive C . It follows then from (2.24), (2.28) – (2.31) that

$$\mathbf{Var}\{\mathcal{N}_n[\varphi]\} \leq (r_{n-1}^{(n)})^2 \left(\sup_{\lambda \in \mathbb{R}} |\varphi'(\lambda)| \right)^2. \quad (2.36)$$

It can be shown [33] that if the potential satisfies the local Lipschitz condition (2.7), then the coefficients $r_{n-1}^{(n)}$ are bounded in n . We conclude that the variance of a linear statistic (1.2) is bounded in n if the test function satisfies (2.35):

$$\mathbf{Var}\{\mathcal{N}_n[\varphi]\} \leq \text{Const} \left(\sup_{\lambda \in \mathbb{R}} |\varphi'(\lambda)| \right)^2. \quad (2.37)$$

This has to be compared with the well known fact of probability theory, according to which the variance of a linear statistics of i.i.d. random variables is $O(n)$ for any test function.

Notice that the condition (2.35) for (2.37) to be valid is pertinent. Indeed, it is shown in Appendix A.1 that for the Gaussian Unitary Ensemble (2.16) and for the piece-wise constant $\varphi_n = \chi_\Delta$ in (1.2), i.e., for (1.1), the variance is $O(\log n)$, as $n \rightarrow \infty$ (see formulas (A.1), (A.10) – (A.11)).

2.2 Asymptotic behavior of $\psi_l^{(n)}(\lambda)$

We present now powerful asymptotic formulas for orthonormal functions $\psi_l^{(n)}$ of (2.21) – (2.24) due to Deift et al [17], valid in the case of a real analytic potential V in (2.1). We give a bit different their form that will be more convenient below. Our form is reminiscent of standard semi-classical formulas and is similar to that, given in [7, 8]. Notice that [8] contains another, although heuristic, derivation of the asymptotic formulas as well as an interesting heuristic explanation of the quasiperiodicity of formulas for $q \geq 2$, based on ideas of statistical mechanics and quantum field theory.

Assume that V is real analytic. Then the support (1.7) of N is a union of $q < \infty$ finite disjoint intervals [17]. The function P in (2.8), (2.11) is also real analytic. Following [17] we say that V is regular if the inequality (2.5) is strict and P is strictly positive on σ . Hence the DOS ρ of (1.6) is strictly positive inside σ and vanishes precisely like a square root at the endpoints. Denote

$$N(\lambda) = N([\lambda, \infty)). \quad (2.38)$$

According to [17], in the regular case there exist functions $d_n(\lambda)$, and $\gamma_n(\lambda)$ such that if λ belongs to the interior of the support (1.7), then

$$\psi_n^{(n)}(\lambda) = (2d_n(\lambda))^{1/2} \cos(\pi n N(\lambda) + \gamma_n(\lambda)) + O(n^{-1}), \quad n \rightarrow \infty. \quad (2.39)$$

Moreover, $d_n(\lambda)$ and $\gamma_n(\lambda)$ depend on n via the vector $n\beta$, where $\beta = (\beta_1, \dots, \beta_{q-1})$ is given by (1.8). This means that there exist n -independent continuous functions $\mathcal{D} : \sigma \times \mathbb{T}^{q-1} \rightarrow \mathbb{R}_+$, and $\mathcal{G} : \sigma \times \mathbb{T}^{q-1} \rightarrow \mathbb{R}$ such that

$$d_n(\lambda) = \mathcal{D}(\lambda, n\beta), \quad \gamma(\lambda) = \mathcal{G}(\lambda, n\beta). \quad (2.40)$$

If λ belongs to the exterior of σ , then $\psi_n^{(n)}(\lambda)$ decays exponentially in n as $n \rightarrow \infty$.

Similar asymptotic formulas are valid for coefficients of the Jacobi matrix $J^{(n)}$ of (2.26), i.e., there exist n -independent continuous functions $\mathcal{R} : \mathbb{T}^{q-1} \rightarrow \mathbb{R}_+$ and $\mathcal{S} : \mathbb{T}^{q-1} \rightarrow \mathbb{R}$ such that

$$r_{n-1}^{(n)} = \mathcal{R}(n\beta) + O(n^{-1}), \quad s_n^{(n)} = \mathcal{S}(n\beta) + O(n^{-1}), \quad n \rightarrow \infty. \quad (2.41)$$

The functions \mathcal{D} , \mathcal{G} , \mathcal{R} , and \mathcal{S} can be expressed via the Riemann theta-function, associated in the standard way with two-sheeted Riemann surface obtained by gluing together two copies of the complex plane slit along the gaps $(b_1, a_2), \dots, (b_{q-1}, a_q), (b_q, a_1)$ of the support of the measure N , the last gap goes through the infinity.

The components of the vector $\beta = \{\beta_l\}_{l=1}^{q-1}$ are rationally independent generically in V , thus the functions $\mathcal{D}(\lambda, n\beta)$, $\mathcal{G}(\lambda, n\beta)$, $\mathcal{R}(n\beta)$, and $\mathcal{S}(n\beta)$ are quasiperiodic in n in general.

We will also need asymptotic formulas for $\psi_{n+k}^{(n)}$ as $n \rightarrow \infty$ and $k = O(1)$ (in particular, we need the case $k = -1$ in (2.31)). They can be extracted from [17] (see also [8]), but it will be convenient to present a different derivation, because the corresponding argument will be used in our analysis of limiting laws for linear eigenvalue statistics. To this end we replace a regular potential V in (2.20) by V/g , $g > 0$, introducing explicitly the amplitude of the potential. Then the quantities of asymptotic formulas (2.38) – (2.41) will depend on g , and it follows from the results of [17, 26] that these quantities will be continuous functions of g in a certain neighborhood of $g = 1$, provided that the support (1.7) for $g = 1$ consists of q disjoint intervals. Consider now $r_{n+k-1}^{(n)}(g)$. Taking into account that the origin of the super-index n in the above formulas is the factor n in front of V in (2.23), we can write

$$n \frac{V}{g} = (n+k) \frac{V}{g(1+k/n)}. \quad (2.42)$$

In other words, to obtain $r_{n+k-1}^{(n)}(g)$ and $\psi_{n+k}^{(n)}(\lambda, g)$ we have to make the change

$$g \rightarrow g + \frac{k}{n}g \quad (2.43)$$

in the inverse amplitude of the potential. We obtain in view of (2.41) for $k = o(n)$:

$$\begin{aligned} r_{n+k-1}^{(n)}(g) &= r_{n+k-1}^{(n+k)}((1+k/n)g) \\ &\simeq \mathcal{R}((1+k/n)g, (n+k)\beta((1+k/n)g)) \\ &\simeq \mathcal{R}(g, n\beta(g) + k\alpha(g)), \end{aligned} \quad (2.44)$$

where

$$\alpha(g) = (g\beta(g))', \quad (2.45)$$

and the symbol " \simeq " denotes here and below the leading term(s) of the corresponding l.h.s. as $n \rightarrow \infty$.

We have an analogous formula for (2.38):

$$(n+k)N(\lambda, (1+k/n)g) \simeq nN(\lambda, g) + k\nu(\lambda, g),$$

where

$$\nu(\lambda, g) = \frac{\partial}{\partial g}(gN(\lambda, g)). \quad (2.46)$$

By using these formulas, we can write for any fixed k (in fact $k = o(n)$):

$$\begin{aligned} \psi_{n+k}^{(n)}(\lambda, g) &\simeq (2\mathcal{D}(\lambda, g, n\beta + k\alpha))^{1/2} \\ &\times \cos\left(\pi nN(\lambda, g) + \pi k\nu(\lambda, g) + \mathcal{G}(\lambda, g, n\beta + k\alpha)\right). \end{aligned} \quad (2.47)$$

Applying to (2.44) and its analog for $s_{n+k}^{(n)}$ the limiting procedure (1.10), we obtain the coefficients

$$r_{k-1}(x) = \mathcal{R}(x + k\alpha), \quad s_k(x) = \mathcal{S}(x + k\alpha), \quad x \in \mathbb{T}^{q-1}, \quad k \in \mathbb{Z}. \quad (2.48)$$

They determine a family of the double infinite Jacobi matrices $J(x)$, $x \in \mathbb{T}^{q-1}$:

$$(J(x)\psi)_k = r_k(x)\psi_{k+1} + s_k(x)\psi_k + r_{k-1}(x)\psi_{k-1}, \quad k \in \mathbb{Z}, \quad (2.49)$$

that can be viewed as a quasiperiodic operator, acting in $l^2(\mathbb{Z})$.

Consider now the functional

$$\mathcal{E}[m] = - \int_{\sigma} \log |\lambda - \mu| m(d\lambda) m(d\mu), \quad (2.50)$$

defined on unit non-negative measures, whose support is contained in σ . This is the standard variational problem of the potential theory [34]. Denote ν the unique minimizer of (2.50). Then, according to [12] (see also [11, 29]), the non-increasing function $\nu(\lambda) = \nu([\lambda, \infty))$ (cf (2.38)) coincides with the function, defined in (2.46). Moreover, according to [31] the measure ν is the Integrated Density of States (IDS) measure of $J(x)$ (see [32] for a definition of the IDS measure in a general setting of ergodic operators), the support (1.7) is spectrum of $J(x)$, and (cf (1.8))

$$\alpha_l = \nu([a_l, \infty)), \quad l = 1, \dots, q-1 \quad (2.51)$$

are the frequencies of quasi-periodic coefficients (2.48) of $J(x)$. In other words, $J(x)$ is a "finite band" Jacobi matrix, well known in spectral theory and integrable systems [41].

By using these facts and also taking into account that $\psi_n^{(n)}(\lambda)$ decays exponentially in n outside the support [17], we can prove that for any continuous $\Phi : \mathbb{R} \rightarrow \mathbb{C}$ of a compact support we have in the limit (1.10) of (2.33)

$$\lim_{n_j(x) \rightarrow \infty} \int_{\mathbb{R}} \Phi(\lambda) e_{n_j(x)+l, n_j(x)+m}^{(n)}(\lambda) d\lambda = \int_{\sigma} \Phi(\lambda) e_{lm}(\lambda, x) d\lambda, \quad (2.52)$$

where we have for $\lambda \in \sigma$:

$$e_{lm}(\lambda, x) = \psi_l^+(\lambda, x) \overline{\psi_m^+(\lambda, x)} + \psi_l^-(\lambda, x) \overline{\psi_m^-(\lambda, x)}, \quad (2.53)$$

$$\psi_l^+(\lambda, x) = e^{i\pi l \nu(\lambda)} \mathcal{U}(\lambda, x + l\alpha), \quad \mathcal{U}(\lambda, x) = (\mathcal{D}(\lambda, x)/2)^{1/2} e^{i\mathcal{G}(\lambda, x)}, \quad (2.54)$$

and $\psi_l^- = \overline{\psi_l^+}$. The above formula can also be written as

$$\psi_l^{\pm}(\lambda, x) = \psi_0^{\pm}(\lambda, x + T^l \alpha), \quad (2.55)$$

where $T : \mathbb{T}^{q-1} \rightarrow \mathbb{T}^{q-1}$ is defined as $Tx = x + \alpha$. This shows that $\Psi^{\pm}(\lambda, x) = \{\psi_l^{\pm}(\lambda, x)\}_{l \in \mathbb{Z}}$ is a generalized (quasi-Bloch) eigenfunction of $J(x)$:

$$J(x)\Psi^{\pm}(\lambda, x) = \lambda\Psi^{\pm}(\lambda, x).$$

In fact, $\{\Psi^{\pm}(\lambda, x)\}_{\lambda \in \sigma}$ form a complete system in $l^2(\mathbb{Z})$.

We refer the reader to [31] for more details of this aspect of asymptotic formulas of [17].

2.3 Asymptotic behavior of variance

We assume in this subsection that the test function φ in (1.2) is of the class C^1 and does not depend on n . Hence, function (2.30) is continuous in (λ_1, λ_2) . As a result, fast oscillating in n functions, entering (2.47), do not contribute to the limit, as was already in obtaining (2.52) – (2.54). We have then from (2.28), (2.34), and (2.52) – (2.54)

$$\mathbf{Var}\{\mathcal{N}_n[\varphi]\} \simeq \mathcal{V}(n\beta), \quad (2.56)$$

where

$$\mathcal{V}(x) = \int_{\sigma} \int_{\sigma} \left| \frac{\Delta\varphi}{\Delta\lambda} \right|^2 \mathcal{V}(\lambda_1, \lambda_2, x) d\lambda_1 d\lambda_2, \quad (2.57)$$

$$\mathcal{V}(\lambda_1, \lambda_2, x) = \mathcal{R}^2(x) (e_{0,0}(\lambda_1, x)e_{-1,-1}(\lambda_2, x) - e_{-1,0}(\lambda_1, x)e_{-1,0}(\lambda_2, x)), \quad (2.58)$$

and $e_{lm}(\lambda, x)$ are given by (2.53) – (2.54), in particular

$$\int_{\sigma} e_{lm}(\lambda, x) d\lambda = \delta_{lm}. \quad (2.59)$$

Since the charges β_l , $l = 1, \dots, q-1$ of (1.8) are continuous and non-constant functionals of the potential (see [17, 26]), the leading term of variance is a quasi-periodic function generically in the potential. In particular, it has no limit as $n \rightarrow \infty$. Its limiting points are indexed by the subset $\mathbb{H}^{q-1} \subset \mathbb{T}^{q-1}$, the closure of limiting points of the $q-1$ dimensional vectors (1.9). \mathbb{H}^{q-1} is \mathbb{T}^{q-1} generically in V , but it can also be a proper subset of \mathbb{T}^{q-1} (see, e.g. (2.15)).

The simplest case is $q = 1$ of a single interval support. Here \mathbb{H}^0 is a point, there exist the limits [2, 16]

$$\lim_{n \rightarrow \infty} r_{n+k-1}^{(n)} = r, \quad \lim_{n \rightarrow \infty} s_{n+k}^{(n)} = s, \quad \forall k \in \mathbb{Z}, \quad (2.60)$$

and the "limiting" Jacobi matrix $J(x)$ of (2.49) has constant coefficients:

$$J_{jk} = r\delta_{j+1,k} + s\delta_{j,k} + r\delta_{j-1,k}.$$

Placing the origin of the spectral axis at s , we obtain that $\lambda = 2r \cos \pi\nu$, ψ^{\pm} are just the plane waves,

$$\sigma = [-2r, 2r],$$

and

$$\mathcal{D}(\lambda) = -\nu'(\lambda) = \frac{1}{\pi\sqrt{4r^2 - \lambda^2}}, \quad \lambda \in \sigma. \quad (2.61)$$

This and general formula (2.58) yield a version of (2.56) – (2.57) in which the role of $\mathcal{V}(x, \lambda_1, \lambda_2)$ plays

$$\mathcal{V}^{(1)}(\lambda_1, \lambda_2) = \frac{1}{4\pi^2} \frac{4r^2 - \lambda_1\lambda_2}{\sqrt{4r^2 - \lambda_1^2}\sqrt{4r^2 - \lambda_2^2}}, \quad \lambda_1, \lambda_2 \in \sigma. \quad (2.62)$$

This form of the variance was first found in physical papers [4, 10] and proved rigorously in [23]. We see that in the single interval case the variance is universal, i.e., its functional form does not depend explicitly on the potential, the information on the potential being encoded in the unique parameter r of (2.60). In particular, we have (2.62) for the Gaussian Unitary Ensemble (2.16) [13].

In the case (2.12) – (2.15) \mathbb{H}^{q-1} consists of q points (see (2.15) and the variance is a q -periodic function of n . Example (2.17) – (2.19) corresponds to the simplest non-trivial case $q = 2$, where $\beta_1 = 1/2$, $\mathbb{H}^1 = \{0, 1/2\}$, the matrix $J(x)$ is 2-periodic, its coefficients are

$$r_k = \frac{b - (-1)^k a}{2}, \quad s_k = 0, \quad (2.63)$$

and the variance is asymptotically 2-periodic function in n [8, 3]:

$$\mathbf{Var}\{\mathcal{N}_n[\varphi]\} \simeq \mathcal{V}^{(2)}(n/2),$$

where $\mathcal{V}^{(2)}(x)$ is given by (2.57) in which $\mathcal{V}(\lambda_1, \lambda_2, x)$, $x \in \mathbb{H}^1 = \{0, 1/2\}$ is

$$\begin{aligned} \mathcal{V}^{(2)}(\lambda_1, \lambda_2, x) &= \frac{1}{2\pi^2} \frac{\varepsilon_{\lambda_1} \varepsilon_{\lambda_2}}{\sqrt{|R_2(\lambda_1)|} \sqrt{|R_2(\lambda_2)|}} \\ &\times ((a^2 - \lambda_1 \lambda_2)(b^2 - \lambda_1 \lambda_2) - (-1)^{2x} ab(\lambda_1 - \lambda_2)^2), \quad \lambda_1, \lambda_2 \in \sigma, \end{aligned} \quad (2.64)$$

with

$$R_2(\lambda) = (\lambda^2 - a^2)(\lambda^2 - b^2), \quad (2.65)$$

and $\varepsilon_\lambda = 1$ if $\lambda \in (-b, -a)$ and $\varepsilon_\lambda = -1$ if $\lambda \in (a, b)$. In fact, these formulas are valid for any real analytic and even potential, producing a symmetric two-interval support

$$\sigma = [-b, -a] \cup [a, b], \quad 0 < a < b < \infty, \quad (2.66)$$

(see [3, 8]). The general case of a two-interval and not necessarily symmetric support was analyzed in [8], where it was found that the variance can be expressed via the classical elliptic functions of Jacobi and Weierstrass.

We conclude that a minimum modification of the limiting law of linear eigenvalue statistics in the case of a multi-interval support of the IDS, comparing with the case of i.i.d. random variables, could be a family of normal laws, indexed by the points of \mathbb{H}^{q-1} . We shall see below that this modification is not sufficient in certain cases.

We remark in conclusion of this section that formulas (2.56) – (2.58) allow us to characterize the universality classes of ensembles (2.1) with respect to the variance in the global regime, i.e., the sets of ensembles (potentials), leading to the same asymptotic form of the variance of linear statistics in the regime. Namely, since the potential is present in (2.56) – (2.58) only via the endpoints (a_1, \dots, b_q) of support and via the charges $(\beta_1, \dots, \beta_{q-1})$ of all but one intervals of the support, these parameters determine a universality class. Notice that the parameters are not necessarily independent.

3 Limiting laws

3.1 Laplace transform of the probability law of linear eigenvalue statistics

In this subsection we obtain an expression for the Laplace transform of the probability law of linear eigenvalue statistics (1.2) via orthogonal polynomials. We consider here real-valued test functions $\varphi : \mathbb{R} \rightarrow \mathbb{R}$. The Laplace transform is evidently

$$Z_n[\varphi] = \mathbf{E}_V \left\{ e^{-\mathring{\mathcal{N}}_n[\varphi]} \right\}, \quad (3.1)$$

where $\mathbf{E}_V \{\dots\}$ denotes the expectation with respect to (2.1) (or (2.20)), determined by a given potential V , and

$$\mathring{\mathcal{N}}_n[\varphi] = \mathcal{N}_n[\varphi] - \mathbf{E}_V \{\mathcal{N}_n[\varphi]\}.$$

It is convenient to introduce the parameter $s \in [0, 1]$ and to consider the function

$$F_n(s) = \log Z_n[s\varphi], \quad s \in [0, 1].$$

It is easy to see that

$$F_n(0) = 0, \quad F'_n(0) = -\mathbf{E}_V \left\{ \mathring{\mathcal{N}}_n[\varphi] \right\} = 0,$$

and

$$F''_n(s) = \mathbf{E}_{V+\frac{s\varphi}{n}} \{\mathcal{N}_n^2[\varphi]\} - \mathbf{E}_{V+\frac{s\varphi}{n}}^2 \{\mathcal{N}_n[\varphi]\} := \mathbf{Var}_{V+\frac{s\varphi}{n}} \{\mathcal{N}_n[\varphi]\}.$$

This yields the following expression for the logarithm of (3.1):

$$\log Z_n[\varphi] = F_n(1) := F_n[\varphi] = \int_0^1 (1-s) \mathbf{Var}_{V+\frac{s\varphi}{n}} \{\mathcal{N}_n[\varphi]\} ds. \quad (3.2)$$

We mention that there exist another expression for the Laplace transform (3.1). It dates back to the Heine formulas in the theory of orthogonal polynomials (see e.g. [40], Theorem 2.1.1) and can be easily obtained from the Gram theorem:

$$\begin{aligned} Z_n[\varphi] &= \det \left\{ \int_{\mathbb{R}} e^{-\mathring{\varphi}(\lambda)} \psi_j^{(n)}(\lambda) \psi_k^{(n)}(\lambda) d\lambda \right\}_{j,k=1}^n \\ &= e^{\mathbf{E}\{\mathcal{N}_n[\varphi]\}} \det(1 - K_{n,\varphi}), \end{aligned}$$

where

$$\mathring{\varphi}(\lambda) = \varphi(\lambda) - \mathbf{E}_V \{\mathcal{N}_n[\varphi]\}, \quad \mathbf{E}_V \{\mathcal{N}_n[\varphi]\} = n \int_{\mathbb{R}} \varphi(\lambda) \mathbf{E}_V \{N_n(d\lambda)\},$$

and $K_{n,\varphi}$ is the integral operator, defined as

$$(K_{n,\varphi}f)(\lambda) = \int_{\mathbb{R}} K_n(\lambda, \mu) (1 - e^{-\varphi(\mu)}) f(\mu) d\mu, \quad \lambda \in \mathbb{R}.$$

These formulas and their analogs for unitary matrices were used to prove various versions of the Central Limit Theorem (see e.g. [6, 23, 24, 37, 38, 39, 42]).

3.2 Asymptotic behavior of the Laplace transform

We will assume in this subsection that φ is real analytic. According to (3.2), (2.28), and (2.31), we have to find the asymptotic form of $\psi_n^{(n)}$ and $\psi_{n-1}^{(n)}$ for the potential $V + s\varphi/n$. We have already seen in the previous section that adding terms of the order $O(n^{-1})$ to the potential we obtain non-trivial contributions to the asymptotic formulas because of fast oscillating in n functions in the r.h.s. of (2.39) – (2.41), etc. The $O(n^{-1})$ terms appeared there because of the passage $n \rightarrow n+k$, leading to (2.45) – (2.48). In this case the terms are proportional to the potential, since we just change its amplitude: $V \rightarrow V(1-k/n)$ (see (2.42) – (2.43)). This required derivatives (2.45) and (2.46) of "frequencies" β_l , $l = 1, \dots, q-1$, and

$N(\lambda)$ of fast oscillating functions in (2.39) – (2.41) with respect to the inverse amplitude g of the potential.

On the other hand, to find the asymptotic behavior of the Laplace transform, we have to add to the potential the term $s\varphi/n$ (see (3.2)). Since $\varphi \neq V$ in general, this requires variational derivatives of frequencies with respect to potential, i.e., we have to add the term $\varepsilon\varphi$ to the potential, and find the derivative of β_l , $l = 1, \dots, q - 1$, and $N(\lambda)$ with respect to ε at $\varepsilon = 0$.

Consider first the case $q = 1$, where the support of the IDS is a single interval. Here the dependence on x of functions $\mathcal{D}, \mathcal{G}, \mathcal{R}$ and \mathcal{S} of (2.39) – (2.41) is absent (see (2.60) – (2.62)). Hence the term $s\varphi/n$ is negligible in the limit $n \rightarrow \infty$, because there are no fast oscillating in n functions in the asymptotics of $\psi_{n+k}^{(n)}$, $k = 0, -1$, $r_{n-1}^{(n)}$ and $s_n^{(n)}$, and we obtain from (3.2) and (2.62):

$$\lim_{n \rightarrow \infty} F_n[\varphi] = \lim_{n \rightarrow \infty} \mathbf{Var}\{\mathcal{N}_n[\varphi]\}/2. \quad (3.3)$$

Notice also that we have here the "genuine" limit as $n \rightarrow \infty$, but not a sublimit (1.10) along a subsequence. We conclude that the Central Limit Theorem is valid in this case. This was proved in [23] by the variational method and for a rather broad class of potentials and test functions (not necessarily real analytic).

As is was shown in the previous section, the variance of a linear statistics with a C^1 test function has no limit as $n \rightarrow \infty$ if $q \geq 2$. Its sublimits are indexed by points of the "hull" $\mathbb{H}^{q-1} \subset \mathbb{T}^{q-1}$. Hence we cannot expect the traditional CLT (3.3), as in the case of $q = 1$. Rather this should to be a collection of the CLT, indexed by \mathbb{H}^{q-1} :

$$\lim_{n_j(x) \rightarrow \infty} F_{n_j(x)}[\varphi] = \lim_{n_j(x) \rightarrow \infty} \mathbf{Var}\{\mathcal{N}_{n_j(x)}[\varphi]\}/2 = \mathcal{V}(x)/2, \quad x \in \mathbb{H}^{q-1}, \quad (3.4)$$

where $\{n_j(x)\}$ and $\mathcal{V}(x)$ are defined in (1.10) and (2.57) – (2.58). We will call this the generalized CLT.

We will show now that the generalized CLT is not always the case for $q \geq 2$. Recall that $N(\lambda)$ and β_l are functionals of V and denote

$$\dot{\beta}_l[\varphi] = \left. \frac{\partial}{\partial \varepsilon} \beta_l \right|_{\varepsilon=0}, \quad l = 1, \dots, q - 1, \quad \dot{N}[\varphi] = \left. \frac{\partial}{\partial \varepsilon} N(\lambda) \right|_{\varepsilon=0} \quad (3.5)$$

the variational derivatives of β_l and $N(\lambda)$ with respect to V . $\dot{\beta}_l[\varphi]$ and $\dot{N}[\varphi]$ are linear functionals of φ and nonlinear functionals of V . It follows from [26] that they are well defined if V is real analytic and regular and φ is real analytic and such that $\max_{\lambda \in \mathbb{R}} |V(\lambda)/\varphi(\lambda)| < \infty$.

Arguing as in Section 2.2, we obtain that in this case $\psi_{n+k}^{(n)}$ is given by (2.47) – (2.48) with the replacement

$$k\alpha_l \rightarrow k\alpha_l + s\dot{\beta}_l[\varphi], \quad \pi k\nu \rightarrow \pi k\nu + \pi s\dot{N}[\varphi].$$

Now, assuming (1.10) and taking into account (3.2), (2.28), and (2.34), we obtain from (3.2):

$$F[\varphi] := \lim_{n_j(x) \rightarrow \infty} \log Z_{n_j(x)}[\varphi] = \int_0^1 (1-s)\mathcal{V}(x + s\dot{\beta}[\varphi])ds, \quad (3.6)$$

where \mathcal{V} is given by (2.57). According to (2.57) \mathcal{V} is a quadratic functional of φ . Hence the functional $F[\varphi]$ is not quadratic in general, because of the presence of the term $s\dot{\beta}[\varphi]$ in the

argument of the integrand in (3.6). In other words we have here a limiting law in the sense of (1.10), but the law is not necessarily Gaussian.

It seems that a general classification of possible cases is rather complex. We thus will give several examples, showing different cases of asymptotic behavior of the Laplace transform of the probability law of linear eigenvalue statistics.

Consider first the case, where the test function is a multiple of the potential:

$$\varphi(\lambda) = tV(\lambda), \quad t \in \mathbb{R}. \quad (3.7)$$

Then (2.45) and the relation $\dot{\beta}_l[V] = -\beta'(g)|_{g=1}$ yield:

$$\dot{\beta}_l[\varphi] = \dot{\beta}_l[tV] = -t(\beta_l(1) - \alpha_l(1)), \quad l = 1, \dots, q-1,$$

where α_l , $l = 1, \dots, q-1$ are defined in (2.51). Hence, if

$$\beta_l(1) = \alpha_l(1), \quad l = 1, \dots, q-1, \quad (3.8)$$

then the integrand in (3.6) does not depend on s , and we obtain in view of (2.56) – (2.57) the generalized Central Limit Theorem (3.4).

The equality (3.8) is valid for any potential of the form (2.12) with $g = 1$ and $v^2 - 4$ having only simple and real zeros, because according to [11]

$$\beta_l(1) = \alpha_l(1) = \frac{q-l}{q}, \quad l = 1, \dots, q-1.$$

It is also valid for any even potential, having two equal local minima and one local maximum, which is high enough to produce a two-interval support (2.66). In this case (3.8) results from the symmetry, implying

$$\beta_1 = \alpha_1 = 1/2 \quad (3.9)$$

(recall that in this case the vectors β of (1.8) and α of (2.45) are one-dimensional: $\beta_1 = N(a)$, $\alpha_1 = \nu(a)$).

In all these cases the limiting Jacobi matrix J of (2.49) is q -periodic ($q = 2$ in the case of (2.66), see (2.63)).

It can also be shown that we have the generalized Central Limit Theorem for potentials (2.12) and $\varphi = tv$ (here the limiting matrix J is also q -periodic).

To demonstrate a possibility to have a non-Gaussian limiting law, we consider a simplest non-trivial case of even potential with the two-interval support (2.66) and of test function

$$\varphi(\lambda) = t\lambda, \quad t \in \mathbb{R}, \quad (3.10)$$

i.e., the case of "linear" linear statistic

$$t \sum_{l=1}^n \lambda_l^{(n)} = t \text{Tr } M_n. \quad (3.11)$$

Since in this case $\Delta\varphi/\Delta\lambda$ of (2.30) is equal to t , it follows from (2.58) and (2.59) that

$$\mathcal{V}(x) = t^2 \mathcal{R}^2(x), \quad (3.12)$$

and then (3.6) implies that in the case (3.10) (and for any support) we have for the exponent of the limiting Laplace transform:

$$F[\varphi] \Big|_{\varphi(\lambda)=t\lambda} = \lim_{n_j(x) \rightarrow \infty} \log Z_{n_j(x)}[\varphi] \Big|_{\varphi(\lambda)=t\lambda} = t^2 \int_0^1 (1-s) \mathcal{R}^2 \left(x + s \beta[\varphi] \Big|_{\varphi(\lambda)=t\lambda} \right) ds. \quad (3.13)$$

According to [8]), it is possible to express the coefficient $\mathcal{R}(x)$, corresponding to the two-interval support, via the Jacobi elliptic function:

$$\mathcal{R}^2(x) = \frac{(b-a)^2}{4} + \frac{ab}{2} \operatorname{cn}^2(x + 1/2), \quad (3.14)$$

where $\operatorname{cn}(x) = \operatorname{cn}(2K(k)x|k)$, $k^2 = 4ab/(a+b)^2$, $K(k)$ is the elliptic integral of the first kind. In view of (3.9) the coefficient r_k of (2.48) is given by (2.63):

$$r_{k-1} = \mathcal{R} \left(\frac{k}{2} \right) = \frac{b - (-1)^k a}{2},$$

and is 2-periodic (see also [2]). In view of (3.12) this implies that the variance of (3.11) is asymptotically 2-periodic in n :

$$\mathcal{R}^2 \left(\frac{n}{2} \right) = \frac{b^2 + a^2}{4} - (-1)^n \frac{ab}{2}. \quad (3.15)$$

Furthermore, it is shown in Appendix that

$$\dot{\beta}_1[\varphi] \Big|_{\varphi(\lambda)=t\lambda} = t\omega, \quad \omega = \frac{a}{4K(a/b)}, \quad (3.16)$$

Hence we obtain from (3.14)

$$F[\varphi] \Big|_{\varphi(\lambda)=t\lambda} = \int_0^t (t-s) \mathcal{R}^2(x + s\omega) ds. \quad (3.17)$$

It follows from (3.16) that ω is irrational generically in a , and b , hence $\mathcal{R}^2(x + s\omega)$ is quasi periodic in s in these cases. Since \mathcal{R}^2 is 1-periodic and real analytic, we can write its Fourier series

$$\mathcal{R}^2(x) = \sum_{m \in \mathbb{Z}} c_m e^{2\pi i m x} \quad (3.18)$$

with fast decaying coefficients. Plugging (3.16), and (3.18) in (3.13), we obtain

$$F[\varphi] \Big|_{\varphi(\lambda)=t\lambda} = \frac{c_0 t^2}{2} - tA'(x) - A(x) + A(x + \omega t), \quad (3.19)$$

where

$$A(x) = \sum_{m \in \mathbb{Z} \setminus \{0\}} \frac{c_m}{(2\pi i m \omega)^2} e^{2\pi i m x}. \quad (3.20)$$

We see that the logarithm of the limiting Laplace transform of the probability law of statistics (3.11) contains not only a multiple of $t^2/2$, that would correspond to the CLT, but also a linear in t term, a constant in t term, and either quasi periodic (generically in a, b , when ω

is irrational) or periodic (in special cases, where ω is rational) function of t . Besides, while the variance of statistics (3.11) is (3.15) in the limit (1.10), the coefficient in front of $t^2/2$ is

$$c_0 = \int_{\mathbb{T}} \mathcal{R}^2(x) dx = \frac{b^2 + a^2}{4},$$

hence is not the variance (3.15).

Notice also that in the case $q \geq 3$ we would have in (3.20) the sum over $\mathbb{Z}^{q-1} \setminus \{0\}$, and the expression

$$\dot{\beta} \cdot m := \dot{\beta}_1 m_1 + \dots + \dot{\beta}_{q-1} m_{q-1},$$

in the denominator, that can be arbitrary small for certain collections of $(m_1, \dots, m_{q-1}) \in \mathbb{Z}^{q-1}$ and $\dot{\beta} := (\dot{\beta}_1, \dots, \dot{\beta}_{q-1}) \in \mathbb{R}^{q-1}$ in the case, where the components of $\dot{\beta}$ are irrational. Hence, to make the series in (3.20) convergent, we have to assume that the components of $\dot{\beta}$ are sufficiently bad approximated by rationals (e.g. a Diophantine condition).

According to Appendix, in a general case of a real analytic φ and a two-interval support

$$\sigma = [a_1, b_1] \cup [a_2, b_2], \quad -\infty < a_1 < b_1 < a_2 < b_2 < \infty, \quad (3.21)$$

we have:

$$\dot{\beta}_1[\varphi] = -\frac{1}{2\pi i I} \int_{\sigma} \frac{\varphi(\mu) - \varphi(-\mu)}{\sqrt{R_2(\mu)}} d\mu, \quad (3.22)$$

where $\sqrt{R_2(\mu)}$ is defined in (2.9),

$$I = \int_{b_1}^{a_2} \frac{d\mu}{\sqrt{(b_2 - \mu)(a_2 - \mu)(\mu - b_1)(\mu - a_1)}} = \frac{2}{((b_2 - b_1)(a_2 - a_1))^{1/2}} K(\kappa), \quad (3.23)$$

and

$$\kappa^2 = \frac{(a_2 - b_1)(b_2 - a_1)}{(b_2 - b_1)(a_2 - a_1)}$$

(see [20], formula (3.149.4)). In the symmetric case (2.66) we have

$$I = \frac{2}{b+a} K\left(\frac{2\sqrt{ab}}{b+a}\right) = \frac{2}{a} K\left(\frac{b}{a}\right),$$

where the second equality results from the formula $(1+k)^{-1}K(2\sqrt{k}/(1+k)) = K(k)$ [20], formula (8.126.3). It follows then that in this case and for $\varphi = t\lambda$ (3.22) coincides with (3.16).

Combining (3.6) for $q = 2$ and (3.22), we obtain the limiting form of the Laplace transform of the probability law of linear eigenvalue statistics for real analytic test functions. In fact, this form is also valid for larger classes of test functions, in particular, for bounded C^1 functions with bounded derivative. Indeed, for any such function φ there exists a sequence $\{\varphi_k\}$ of real analytic test functions, converging to φ in the C^1 metrics. Besides:

- (i) \mathcal{V} of (2.57) is continuous in φ in the C^1 metrics and in $x \in \mathbb{T}^1$;
- (ii) $\dot{\beta}_1$ of (3.22) is continuous in φ in the C metrics;

(iii) we have for the functional F_n of (3.2)

$$|F_n[\varphi_1] - F_n[\varphi_2]| \leq C \left(\sup_{\lambda \in \mathbb{R}} |\varphi'_1 - \varphi'_2| \right)^2, \quad (3.24)$$

where C depend only on the potential and is finite if (2.2) holds. To prove this inequality, implying the uniform in n the C^1 continuity of F_n in φ , we use an argument, similar to that proving (3.2), to obtain

$$F[\varphi_1] - F[\varphi_2] = \int_0^1 (1-s) \text{Var}_{V+\varphi_s/n} \{ \mathcal{N}_n[\varphi_1 - \varphi_2] \} ds,$$

where $\varphi_s = s\varphi_1 + (1-s)\varphi_2$, $s \in [0, 1]$. This and (2.37) yield (3.24).

Now we can write:

$$|F_{n_j(x)}[\varphi] - F_x[\varphi]| \leq |F_{n_j(x)}[\varphi] - F_{n_j(x)}[\varphi_k]| + |F_{n_j(x)}[\varphi] - F_x[\varphi_k]| + |F_x[\varphi] - F_x[\varphi_k]|$$

for F_n of (3.2) and F_x of (3.6) with explicitly indicated dependence on $x \in \mathbb{T}^1$. Making the limit $n_j(x) \rightarrow \infty$ and using (3.24), we obtain in view of the above result on real analytic test functions:

$$\lim_{n_j(x) \rightarrow \infty} \sup |F_{n_j(x)}[\varphi] - F_x[\varphi]| \leq C \left(\sup_{\lambda \in \mathbb{R}} |\varphi' - \varphi'_k| \right)^2 + |F_x[\varphi] - F_x[\varphi_k]|.$$

Now the C^1 limit $\varphi_k \rightarrow \varphi$ and above assertions (i) and (ii) yield the validity of our results (3.6) for $q = 2$ and (3.22) for bounded C^1 test functions with bounded derivative, hence a possibility to have non-Gaussian limiting laws (the non-validity of the Central Limit Theorem) for linear eigenvalue statistics of ensembles (2.1) with these test functions.

Formula (3.22) allows us to characterize the class of potentials and test functions for which the (generalized) Central Limit Theorem (3.4) is valid in the case of a general two-interval support (3.21). Indeed, for any pair (V, φ) for which the r.h.s. of (3.22) is zero the integrand of (3.6) does not depend on s and we have the generalized CLT (3.4). In particular, in the symmetric case (2.66) it follows from (3.22) that $\beta_1[\varphi]$ is zero if and only if

$$\int_a^b \frac{\varphi(\mu) - \varphi(-\mu)}{\sqrt{(b^2 - \mu^2)(\mu^2 - a^2)}} d\mu = 0,$$

In particular, for an even potential of support (2.66) and an even test function φ the generalized CLT is valid.

One can view φ as an analog of external field in statistical mechanics. Hence, we can say that in this case an even external field "does not break the symmetry". On the other hand a "generic" φ or an odd φ , such that

$$\int_a^b \frac{\varphi(\mu) d\mu}{\sqrt{(b^2 - \mu^2)(\mu^2 - a^2)}} \neq 0,$$

is a "breaking symmetry field" and leads to a non-Gaussian limiting law. Its simplest case $\varphi(\lambda) = t\lambda$ (3.10) is given by (3.19).

4 Intermediate and local regimes

In this section we consider limiting laws of linear eigenvalue statistics for the test functions, given by (1.11) and (1.12) with a C^1 function φ and λ_0 belonging to the interior of σ .

We begin again by calculating the asymptotic form of the variance in these cases. Changing variables to

$$\lambda_{1,2} = \lambda_0 + t_{1,2}/n^\alpha, \quad 0 < \alpha \leq 1, \quad (4.1)$$

we obtain from (2.28):

$$\mathbf{Var}\{\mathcal{N}_n[\varphi]\} = \int_{\mathbb{R}} \int_{\mathbb{R}} \left(\frac{\Delta\varphi}{\Delta t} \right)^2 \mathcal{V}_n(\lambda_0 + t_1/n^\alpha, \lambda_0 + t_2/n^\alpha) dt_1 dt_2. \quad (4.2)$$

To find the asymptotic form of the r.h.s. we will use again (2.34) and (2.33) in which $\psi_{n+k}^{(n)}(\lambda)$, $k = 0, -1$ is given by (2.47) with $\lambda = \lambda_0 + t/n^\alpha$. Taking into account that \mathcal{D}, N , and \mathcal{G} are smooth functions of λ in a sufficiently small neighborhood of λ_0 , we can write

$$\begin{aligned} \psi_{n+k}^{(n)}(\lambda) &\simeq (2\mathcal{D}(\lambda_0, n\beta + k\alpha))^{1/2} \\ &\times \cos(\pi n N(\lambda_0) + \pi k \nu(\lambda_0) - \pi \rho(\lambda_0) n^{1-\alpha} + \mathcal{G}(\lambda_0, n\beta + k\nu)), \end{aligned}$$

where $\rho(\lambda) = -N'(\lambda)$, and we do not indicate the dependence on g (in fact it suffices to consider the case $g = 1$). Plugging this into (2.33) and (2.34), and omitting in the resulting integrand of the r.h.s. of (4.2) the fast oscillating terms, we obtain

$$\begin{aligned} \mathbf{Var}\{\mathcal{N}_n[\varphi]\} &\simeq B(\lambda_0, n\beta) \\ &\times \int_{\mathbb{R}} \int_{\mathbb{R}} (\varphi(t_1) - \varphi(t_2))^2 \frac{\sin^2(\pi \rho(\lambda_0)(t_1 - t_2)n^{1-\alpha})}{2\pi^2(t_1 - t_2)^2} dt_1 dt_2, \end{aligned} \quad (4.3)$$

where

$$\begin{aligned} B(\lambda, x) &= 2\pi^2 \mathcal{R}^2(x) \mathcal{D}(\lambda, x) \mathcal{D}(\lambda, x - \alpha) \\ &\times \sin^2(\pi \nu(\lambda) + \mathcal{G}(\lambda, x) - \mathcal{G}(\lambda, x - \alpha)). \end{aligned} \quad (4.4)$$

This leads to the following result in the local regime $\alpha = 1$ and for the limit (1.10):

$$\begin{aligned} \lim_{n_j(x) \rightarrow \infty} \mathbf{Var}\{\mathcal{N}_{n_j(x)}[\varphi_{n_j(x)}]\} &= B(\lambda_0, x) \\ &\times \int_{\mathbb{R}} \int_{\mathbb{R}} (\varphi(t_1) - \varphi(t_2))^2 \frac{\sin^2(\pi \rho(\lambda_0)(t_1 - t_2))}{2\pi^2(t_1 - t_2)^2} dt_1 dt_2. \end{aligned} \quad (4.5)$$

It is known that in the local regime the variance of linear eigenvalue statistics has a universal limiting form [17, 33]

$$\lim_{n \rightarrow \infty} \mathbf{Var}\{N_n[\varphi_n]\} = \int_{\mathbb{R}} \int_{\mathbb{R}} (\varphi(t_1) - \varphi(t_2))^2 \frac{\sin^2(\pi \rho(\lambda_0)(t_1 - t_2))}{2\pi^2(t_1 - t_2)^2} dt_1 dt_2, \quad (4.6)$$

in which all the information on the potential is encoded in $\rho(\lambda_0)$. This and (4.5) imply the identity

$$B(\lambda, x) = 1, \quad (4.7)$$

valid for all $x \in \mathbb{T}^{q-1}$ and λ , belonging to the interior of the σ (a direct proof of the identity can be extracted from the proof of Lemma 6.1 of [17]).

In the intermediate regime $0 < \alpha < 1$ we still have a fast oscillating factor

$$\sin^2(n^{1-\alpha}\pi\rho(\lambda_0)(t_1 - t_2))$$

in the integrand of (4.3). Replacing the factor by its average 1/2, and using (4.7) we obtain in this regime

$$\lim_{n \rightarrow \infty} \mathbf{Var}\{N_n[\varphi_n]\} = \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{(\varphi(t_1) - \varphi(t_2))^2}{4\pi^2(t_1 - t_2)^2} dt_1 dt_2. \quad (4.8)$$

As in the case of the local regime the limit here is the same for any subsequence (1.10), hence we do not need to assume (1.10). We conclude that the variance of linear statistics has a well defined limit in the intermediate regime as well. Moreover, (4.8) is the "smoothed" version of variance (4.6) in the local regime, since (4.8) is (4.6) in which the "oscillating" factor $\sin^2(\pi\rho(\lambda_0)(t_1 - t_2))$ is replaced by its average 1/2.

Passing to the Fourier transform

$$\widehat{\varphi}(k) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{ikt} \varphi(t) dt,$$

we can rewrite the r.h.s. of (4.8) in the form

$$\int_{\mathbb{R}} |k| \widehat{\varphi}(k) \widehat{\varphi}(-k) dk$$

appearing in the continuous analog of the strong Szegő theorem (see [6, 18, 37] and references therein).

The universality property of unitary invariant Matrix Models [17, 33] implies that the Laplace transform of the probability law of linear eigenvalue statistics has the following limiting form in the local regime

$$\lim_{n \rightarrow \infty} Z_n[\varphi_n] \Big|_{\varphi_n(\lambda) = \varphi((\lambda - \lambda_0)n)} = e^{2\pi\rho(\lambda_0)\widehat{\varphi}(0)} \det(1 - S_\varphi), \quad (4.9)$$

where S_φ is the integral operator, defined as

$$(S_\varphi f)(t) = \int_{\mathbb{R}} \frac{\sin \pi\rho(\lambda_0)(t - u)}{\pi\rho(\lambda_0)(t - u)} (1 - e^{-\varphi(u)}) f(u) du, \quad (4.10)$$

and we assume that φ in (1.12) is continuous and integrable on \mathbb{R} . It is obvious that the logarithm of the r.h.s. of (4.9) is not quadratic in φ , hence the CLT is not valid in the local regime (see e.g. [22] for related results).

If, however, we take in the above formulas

$$\varphi(t) = \Phi((t - t_0)\delta),$$

where Φ does not depend on $\delta \rightarrow 0$, and $t_0 \in \mathbb{R}$, i.e., we assume that the test function in (4.9) – (4.10) is "slow varying", then it can be shown (see e.g. [6, 15, 39]) that the limit of the r.h.s. of (4.9) as $\delta \rightarrow 0$ is the r.h.s. of (4.8), divided by 2.

On the other hand, take as a test function in (3.6)

$$\varphi(\lambda) = \Phi((\lambda - \lambda_0)/\delta), \quad (4.11)$$

where Φ does not depend on $\delta \rightarrow 0$, and λ_0 belongs to the interior of the support of N , i.e., assume that φ is "fast varying". Since the variational derivatives (linear functionals of φ) $\dot{\beta}_l[\varphi]$, $l = 1, \dots, q - 1$ of (3.5) can be written as

$$\dot{\beta}_l[\varphi] = \int_{\mathbb{R}} b_l(\lambda) \varphi(\lambda) d\lambda,$$

we have

$$\dot{\beta}_l[\varphi] = \delta \int_{\mathbb{R}} b_l(\lambda_0 + \delta t) \Phi(t) dt \rightarrow 0, \quad \delta \rightarrow 0.$$

Hence the term $s\dot{\beta}$ in the argument of the integrand of (3.6) vanishes in the limit $\delta \rightarrow 0$ and we obtain from (2.57), changing variables to $\lambda_{1,2} = \lambda_0 + \delta t_{1,2}$:

$$\lim_{\delta \rightarrow \infty} F[\varphi] \Big|_{\varphi(\lambda) = \Phi((\lambda - \lambda_0)/\delta)} = \frac{1}{2} \mathcal{V}(x), \quad (4.12)$$

where

$$\mathcal{V}(x) = \mathcal{V}(\lambda_0, \lambda_0, x) \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{(\Phi(t_1) - \Phi(t_2))^2}{2\pi^2(t_1 - t_2)^2} dt_1 dt_2.$$

Now it can be shown, by using (2.58), (2.54), (2.55), and (4.4), that the r.h.s. of the last formula coincides with (4.8). Hence the limit (4.12) coincides again with the r.h.s. of (4.8), divided by 2.

The above suggests that the CLT is valid in the intermediate regime. This was indeed proved in several particular cases (see [37] and references therein).

A Appendices

A.1 Variance of the Eigenvalue Counting Measure of the GUE

It has been shown in Sections 2.1 and 2.3 that the variance of linear eigenvalue statistics is of the order $O(1)$ if the corresponding test function is of the class C^1 . Here we will argue that this condition is close to be optimal. To this end we will show that linear eigenvalue statistics for the GUE (2.16) are of the order $O(\log n)$ if φ is the indicator χ_{Δ} of an interval $\Delta = (a, b) \in (-2w, 2w)$. For this choice of φ the corresponding linear statistic is the Eigenvalue Counting Measure (1.1). We will derive the asymptotic formula

$$\mathbf{Var}\{\mathcal{N}_n(\Delta)\} = \frac{1}{\pi^2} \log n (1 + o(1)), \quad n \rightarrow \infty. \quad (A.1)$$

We have from (2.27)

$$\mathbf{Var}\{\mathcal{N}_n(\Delta)\} = \int_a^b d\lambda \int_{\mathbb{R} \setminus (a,b)} K_n^2(\lambda, \mu) d\mu, \quad (A.2)$$

where K_n is given by (2.29). According to (2.32) we need asymptotics of $\psi_l^{(n)}$ for $l = n-1, n$ to find the asymptotic behavior of (A.2). These follow from (2.47) (with the remainder $o(1)$), (2.61), and (2.16) or can be obtained from the well known Plancherel-Rotah formulas for the Hermite polynomials (see [40], formula (8.22.12)):

$$\psi_{n+k}^{(n)}(\lambda) = \frac{1}{(2\pi\sqrt{g}\sin\theta)^{1/2}} \cos(n\alpha(\theta) + k\theta + \mathcal{G}(\theta)) (1 + o(1)), \quad (\text{A.3})$$

where $\lambda = 2\sqrt{g}\cos\theta$, $\alpha(\theta) = \theta - \sin 2\theta/2$, $\mathcal{G}(\theta) = \alpha(\theta)/2 - \pi/4$.

Besides, in this case [40]

$$r_{n-1}^{(n)} = \sqrt{g}. \quad (\text{A.4})$$

We write (A.2) as

$$\mathbf{Var}\{\mathcal{N}_n(\Delta)\} = I_1 + I_2 + I_3 + I_4, \quad (\text{A.5})$$

where

$$\begin{aligned} I_1 &= \int_{b-a}^{\infty} dx \int_0^{b-a} K_n^2(b-y, b-y+x) dy, \\ I_2 &= \int_0^{b-a} dx \int_0^x K_n^2(b-y, b-y+x) dy, \end{aligned} \quad (\text{A.6})$$

and I_3 and I_4 can be obtained from I_1 and I_2 by replacing $K_n(b-y, b-y+x)$ by $K_n(a+y, a+y-x)$.

By using the Christoffel-Darboux formula (2.32) we can write the integrands in $I_{1,2}$ as

$$(r_{n-1}^{(n)})^2 \Psi_n^2(x, y) x^{-2},$$

where

$$\Psi_n(x, y) = \left(\psi_n^{(n)}(\lambda) \psi_{n-1}^{(n)}(\mu) - \psi_{n-1}^{(n)}(\lambda) \psi_n^{(n)}(\mu) \right) \Big|_{\lambda=b-y, \mu=b-y+x}. \quad (\text{A.7})$$

In view of (A.4) the integral I_1 is bounded from above by

$$g(b-a)^{-2} \int_{\mathbb{R}^2} \left(\psi_n^{(n)}(\lambda) \psi_{n-1}^{(n)}(\mu) - \psi_{n-1}^{(n)}(\lambda) \psi_n^{(n)}(\mu) \right)^2 d\lambda d\mu = 2g/(b-a)^2,$$

where in writing the last equality we used the orthonormality of the system $\{\psi_l^{(n)}\}_{l \geq 0}$.

To find the asymptotic behavior of I_2 we will use formula (A.3). However its direct use is impossible since the remainder term leads to the divergent integral in x . Thus, we write this integral as the sum of the two, over the intervals $(0, A/n)$ and $(A/n, b-a)$, where A is fixed. In the first integral we write (A.7) as

$$\Psi_n(x, y) = \left(\psi_n^{(n)}(\lambda) - \psi_n^{(n)}(\mu) \right) \psi_{n-1}^{(n)}(\mu) - \left(\psi_{n-1}^{(n)}(\lambda) - \psi_{n-1}^{(n)}(\mu) \right) \psi_n^{(n)}(\mu) \Big|_{\lambda=b-y, \mu=b-y+x}.$$

In view of (A.3) $\psi_{n+k}^{(n)}$, $k = 0, -1$ are of order $O(1)$. Besides, we have the relation [40]

$$\frac{d}{d\lambda} \psi_l^{(n)} = -\frac{n}{g} \lambda \psi_l^{(n)} + \left(\frac{nl}{g} \right)^{1/2} \psi_{l-1}^{(n)},$$

implying that $\frac{d}{d\lambda}\psi_{n+k}^{(n)}$ is $O(n)$, $k = 0, -1$. This yields the bound $|\Psi_n(x, y)| \leq \text{const} \cdot nx$, according to which the first integral is

$$\text{const} \cdot n^2 \int_0^{A/n} x dx = O(1), \quad n \rightarrow \infty.$$

In the integral over $(A/n, b-a)$ we use the asymptotic formula (2.47), neglecting, as it was done in the proof of the previous sections, the fast oscillating terms, and noticing that the remainder term $o(1)$ in (2.47) yields the error $o(\log n)$. This and (A.4) for $l = n$ lead to the asymptotic expression (cf 2.62)

$$\frac{1}{2\pi^2} \int_{A/n}^{b-a} \frac{dx}{x^2} \int_0^x dy \frac{4g - \lambda\mu}{\sqrt{4g - \lambda^2}\sqrt{4g - \mu^2}} \Big|_{\lambda=b-y, \mu=b-y+x} + o(1)$$

The leading contribution to the integral is due the integral in x over an interval $(A/n, \varepsilon)$, where $\varepsilon > 0$ is small enough and n -independent, because the integral over $(\varepsilon, b-a)$ is n -independent, hence is $O(1)$, $n \rightarrow \infty$. Then the condition $\lambda = b - y$, $\mu = b - y + x$ can be replaced by $\lambda = b - y$, $\mu = b - y$. As a result the integral in y is asymptotically x , and we obtain

$$I_1 + I_2 = \frac{1}{2\pi^2} \log n + O(1), \quad n \rightarrow \infty.$$

The same contribution is due the sum $I_3 + I_4$ in (A.5), hence we obtain (A.1).

Analogous formula is also known for unitary matrices. For this and corresponding Central Limit Theorem see [24, 42].

Consider the covariance $\mathbf{Cov}\{\mathcal{N}_n(\Delta_1), \mathcal{N}_n(\Delta_2)\}$ of the Eigenvalue Counting Measures (1.1) for $\Delta_1 = (a_1, b_1)$ and $\Delta_2 = (a_2, b_2)$, $\Delta_{1,2} \subset (-2\sqrt{g}, 2\sqrt{g})$. It follows from (2.20) that (cf (2.27))

$$\begin{aligned} \mathbf{Cov}\{\mathcal{N}_n[\varphi_1], \mathcal{N}_n[\varphi_2]\} &:= \mathbf{E}\{\mathcal{N}_n[\varphi_1]\mathcal{N}_n[\varphi_2]\} - \mathbf{E}\{\mathcal{N}_n[\varphi_1]\}\mathbf{E}\{\mathcal{N}_n[\varphi_2]\} \\ &= \frac{1}{2} \int_{\mathbb{R}} \int_{\mathbb{R}} \left(\varphi_1(\lambda_1) - \varphi_1(\lambda_2) \right) \left(\varphi_2(\lambda_1) - \varphi_2(\lambda_2) \right) K_n^2(\lambda_1, \lambda_2) d\lambda_1 d\lambda_2. \end{aligned}$$

This yields for $\varphi_{1,2} = \chi_{\Delta_{1,2}}$

$$\mathbf{Cov}\{\mathcal{N}_n(\Delta_1), \mathcal{N}_n(\Delta_2)\} = \int_{\mathbb{R}^2} [\chi_{(\Delta_1 \cap \Delta_2) \times \mathbb{R}}(\lambda, \mu) - \chi_{\Delta_1 \times \Delta_2}(\lambda, \mu)] K_n^2(\lambda, \mu) d\lambda d\mu. \quad (\text{A.8})$$

The argument, proving (A.1), allows us to find the leading terms of the covariance as $n \rightarrow \infty$ in various cases. We will assume below for the sake of definiteness that $a_1 < b_1$, $a_2 < b_2$, $b_1 \leq b_2$, and $b_1 - a_1 \leq b_2 - a_2$, and consider the following cases of the asymptotic behavior of (A.8) (see [42] for analogous results for unitary matrices).

(i). Disjoint intervals ($b_1 < a_2$):

$$-\frac{1}{4\pi^2} \int_{\Delta_1} d\lambda \int_{\Delta_2} \frac{4g - \lambda\mu}{\sqrt{4g - \lambda^2}\sqrt{4g - \mu^2}} d\mu, \quad (\text{A.9})$$

i.e., in this case the covariance is bounded as it was for the case of linear statistics, generated by C^1 functions in (2.37) and (2.56).

(ii). Touching (outside) intervals ($a_2 = b_1$):

$$-\frac{1}{2\pi^2} \log n. \quad (\text{A.10})$$

(iii) Touching (inside) intervals ($a_1 = a_2$, but $b_1 < b_2$):

$$\frac{1}{2\pi^2} \log n. \quad (\text{A.11})$$

(iv) Embedded intervals ($a_2 < a_1$, $b_1 < b_2$):

$$\frac{1}{4\pi^2} \int_{\Delta_1} d\lambda \int_{\mathbb{R} \setminus \Delta_2} \frac{4g - \lambda\mu}{\sqrt{4g - \lambda^2} \sqrt{4g - \mu^2}} d\mu. \quad (\text{A.12})$$

(v) Intersecting intervals ($a_2 < b_1$ but $a_1 < a_2$ and $b_1 < b_2$). In this case we can write the equality

$$\chi_{(\Delta_1 \cap \Delta_2) \times \mathbb{R}} - \chi_{\Delta_1 \times \Delta_2} = \chi_{(a_1, a_2) \times (\mathbb{R} \setminus (a_1, b_2))} - \chi_{(a_1, a_2) \times (b_1, b_2)}, \quad (\text{A.13})$$

showing that the domain of integration in (A.8) does not contain the “dangerous” diagonal $\lambda = \mu$. Hence, the leading term of the covariance in this case is of the order $O(1)$, and the respective coefficient is given by the integral of the product of the integrand of (A.12) and of (A.13).

By using similar reasoning, it is possible to find the asymptotic form of the variance and the covariance of the Eigenvalue Counting Measure in other cases. Note that in the “regular” cases, where the covariance is bounded, (see (2.56) and (A.9)), its leading term is additive in Δ_1 and Δ_2 (or in φ_1 and φ_2), while in the “singular” cases, where $\mathbf{Cov}\{\mathcal{N}_n(\Delta_1), \mathcal{N}_n(\Delta_2)\} = O(\log n)$, its leading term is independent of Δ_1 and Δ_2 .

A.2 Variational derivative of frequency in the two-interval case

We derive here formula (3.16). We will use the variational approach, based on the functional (2.3).

Write the minimum condition (2.4), (2.6) for $V + \varepsilon\varphi$, and compute its derivative at $\varepsilon = 0$. This yields

$$\varphi(\lambda) - 2 \int_{\sigma} \log |\lambda - \mu| \dot{\rho}(\mu) d\mu = \text{const}, \quad \lambda \in \sigma, \quad (\text{A.14})$$

where

$$\dot{\rho} = \left. \frac{\partial}{\partial \varepsilon} \rho \right|_{\varepsilon=0},$$

and ρ is the density of the measure N of (1.5). Notice that the differentiation of the limits of integration in (2.6) does not contribute to (A.14), because ρ vanishes at each endpoints of the support according to (2.8).

The derivative of (A.14) in λ is the singular integral equation (cf (2.10)):

$$\text{v.p.} \int_{\sigma} \frac{\dot{\rho}(\mu) d\mu}{\mu - \lambda} = -\frac{\varphi'(\lambda)}{2}, \quad \lambda \in \sigma.$$

The general solution of the equation in the case (2.66) is [28]

$$\frac{C_1\lambda + C}{X_2(\lambda)} + \frac{1}{2\pi^2 X_2(\lambda)} \text{v.p.} \int_{\sigma} \frac{\varphi'(\mu) X_2(\mu) d\mu}{\mu - \lambda},$$

where

$$X_2(\lambda) = -i\sqrt{R_2(\lambda)}, \quad \sqrt{R_2(\lambda)} = \sqrt{R_2(z)} \Big|_{z=\lambda+i0},$$

$R_2(z) = (z^2 - a^2)(z^2 - b^2)$ (see (2.9) for $q = 2$) and $\sqrt{R_2(z)}$ is the branch of the square root, fixed by the condition: $\sqrt{R(z)} = z^2 + O(z)$, $z \rightarrow \infty$. The branch assumes pure imaginary values of opposite sides on the edges of σ , seen as a cut of \mathbb{C} .

Taking into account the equalities

$$\int_{\sigma} \dot{\rho}(\mu) d\mu = 0,$$

and

$$\int_{\sigma} \frac{d\mu}{\sqrt{R_2(\mu)}} = 0, \quad \int_{\sigma} \frac{\mu d\mu}{\sqrt{R_2(\mu)}} = -\pi i,$$

we find that

$$\dot{\rho}(\lambda) = \frac{iC}{\sqrt{R_2(\lambda)}} + \frac{1}{2\pi^2 \sqrt{R_2(\lambda)}} \text{v.p.} \int_{\sigma} \frac{\varphi'(\mu) \sqrt{R_2(\mu)} d\mu}{\mu - \lambda}. \quad (\text{A.15})$$

The constant C can be found as follows. Denote $f(z)$ the Stieltjes transform of ρ :

$$f(z) = \int_{\sigma} \frac{\rho(\mu) d\mu}{\mu - z}, \quad z \notin \sigma. \quad (\text{A.16})$$

Recalling that V is real analytic, using (2.8), (2.11), and arguing as above, we find that

$$f(z) = -\frac{V'(z)}{2} - \frac{\sqrt{R_2(z)}}{2\pi i} \int_{\sigma} \frac{V'(\mu) - V'(z)}{\mu - z} \frac{d\mu}{\sqrt{R_2(\mu)}} \quad (\text{A.17})$$

(the analog of the formula with R_q instead of R_2 is valid for any finite number q of intervals in (1.7)).

Write now the minimum condition (2.4) as

$$V_{eff}(b) - V_{eff}(a) = 0,$$

or, in view of (2.6) and (A.16), as

$$\int_{-a}^a \left(f(\lambda) + \frac{V'(\lambda)}{2} \right) d\lambda = 0. \quad (\text{A.18})$$

The condition is valid for any potential, in particular, for $V + \varepsilon\varphi$. According to (A.17) for any sufficiently small ε the integrand here is proportional to $\sqrt{R_2(\lambda)}$, in which a and b are now functions of ε . Hence, the integrand vanishes at the edges of the support and the derivative of (A.18) with V replaced by $V + \varepsilon\varphi$ with respect to ε at $\varepsilon = 0$ is

$$\int_{-a}^a \left(\dot{f}(\lambda) + \frac{\varphi'(\lambda)}{2} \right) d\lambda = 0. \quad (\text{A.19})$$

This and the formula

$$\dot{f}(z) = \int_{\sigma} \frac{\dot{\rho}(\mu)d\mu}{\mu - z} = -\frac{\pi C}{\sqrt{R_2(z)}} - \frac{1}{2\pi i \sqrt{R_2(z)}} \int_{\sigma} \frac{\varphi'(\mu)\sqrt{R_2(\mu)}}{\mu - z} d\mu, \quad (\text{A.20})$$

following from (A.15), yield in the case $\varphi(\lambda) = t\lambda$:

$$\dot{\rho}(\lambda) = \frac{C}{\sqrt{(b^2 - \lambda^2)(\lambda^2 - a^2)}} + \frac{P_2(\lambda)}{2\pi\sqrt{(b^2 - \lambda^2)(\lambda^2 - a^2)}}, \quad \lambda \in \sigma,$$

where $P_2(\lambda) = \lambda^2 - (a^2 + b^2)/2$,

$$C = \frac{tI_2}{2\pi I_1}, \quad (\text{A.21})$$

and

$$I_1 = \int_{-a}^a \frac{d\lambda}{\sqrt{(b^2 - \lambda^2)(a^2 - \lambda^2)}}, \quad I_2 = 2 \int_{-a}^a \frac{P_2(\lambda)d\lambda}{\sqrt{(b^2 - \lambda^2)(a^2 - \lambda^2)}}. \quad (\text{A.22})$$

Now we can find $\dot{\beta}_1[\varphi]$ for $\varphi(\lambda) = t\lambda$. We have by (A.21) and (A.22):

$$\dot{\beta}_1 := \int_a^b \dot{\rho}(\lambda)d\lambda = \frac{t}{2\pi I_1}(I_2 J_1 - I_1 J_2), \quad (\text{A.23})$$

where

$$J_1 = \int_a^b \frac{d\lambda}{\sqrt{(b^2 - \lambda^2)(\lambda^2 - a^2)}}, \quad J_2 = \int_a^b \frac{P_2(\lambda)d\lambda}{\sqrt{(b^2 - \lambda^2)(\lambda^2 - a^2)}}.$$

By using standard formulas (see e.g. [20], formulas (3.159)), we find

$$I_1 = 2K(k)/a, \quad I_2 = 2a \left(\frac{1 - k^2}{2} K(k) - E(k) \right), \quad (\text{A.24})$$

$$J_1 = K(k')/a, \quad J_2 = a \left(E(k') - \frac{1 + k^2}{2} K(k') \right), \quad (\text{A.25})$$

where $K(k)$ and $E(k)$ are the complete elliptic integrals of the first and second kind, $k = b/a$, $k' = \sqrt{1 - k^2}$. These formulas and the identity $EK' + E'K - KK' = \pi/2$, where $K' = K(k')$, $E' = E(k')$ ([20], formula (8.122)) imply (3.16).

A more involved version of the above argument leads to (3.22). We note first that to prove the formula for a real analytic φ it suffices to consider

$$\varphi_{z_0}(\lambda) = \frac{1}{\lambda - z_0}, \quad z_0 \notin \sigma.$$

We have in this case (see below):

$$\dot{\beta}_1[\varphi_{z_0}] = -\frac{1}{2I\sqrt{R_2(z_0)}}, \quad (\text{A.26})$$

where I is defined in (3.23).

Assuming that this formula is valid and using the Cauchy theorem to write

$$\varphi(\lambda) = \frac{1}{2\pi i} \int_{C_{\sigma}} \frac{\varphi(z_0)dz_0}{\lambda - z_0}, \quad \lambda \in \sigma,$$

where C_σ is the contour encircling σ in the clockwise direction, we obtain in view of the linearity of $\dot{\beta}_1[\varphi]$ in φ :

$$\begin{aligned}\dot{\beta}_1[\varphi] &= \frac{1}{2\pi i} \int_{C_\sigma} \dot{\beta}_1[\varphi_{z_0}]\varphi(z_0)dz_0 \\ &= -\frac{1}{2I} \frac{1}{2\pi i} \int_{C_\sigma} \frac{\varphi(z_0)}{\sqrt{R_2(z_0)}} dz_0.\end{aligned}$$

Now the relation $\sqrt{R_2(\lambda - i0)} = -\sqrt{R_2(\lambda + i0)}$ yields (3.22).

To prove (A.26) we use the general formulas (A.15), (A.20), and (A.19) with

$$\varphi'_{z_0}(\lambda) = -\frac{1}{(\lambda - z_0)^2} = -\frac{\partial}{\partial z_0} \frac{1}{\lambda - z_0}.$$

Arguing as in the case $\varphi(\lambda) = t\lambda$ above, we obtain an analog of (A.23) whose denominator contains I of (3.23) instead of I_1 of (A.22), and whose numerator is a bilinear combination of integrals of $|R_2(\mu)|^{-1/2}$ over $[a_2, b_2]$ and of derivatives with respect to z_0 of the integrals over $[b_1, a_2]$ and $[a_2, b_2]$ of $[(\mu - z_0)|R_2(\mu)|^{1/2}]^{-1}$ and $P_2(\mu)[(\mu - z_0)|R_2(\mu)|^{1/2}]^{-1}$, where P_2 is defined now by the relation $\sqrt{R_2(z)} = P_2(z) + O(1/z)$, $z \rightarrow \infty$. These integrals can be expressed via the complete elliptic integrals of the first, second, and third kinds. Furthermore, the complete elliptic integrals of the third kind can be expressed via the incomplete elliptic integrals of the first and the second kinds, whose arguments depend on z_0 . This allows us to obtain a formula for $\dot{\beta}_1[\varphi_{z_0}]$, whose numerator is expressed via the complete elliptic integrals of the first and the second kind and derivatives with respect to z_0 of the incomplete elliptic integrals of the first and the second kind. The derivatives are proportional to $(R_2(z_0))^{-1/2}$ (see [20], formulas (8.123)) This and a bit tedious algebra lead eventually to (A.26).

Another derivation of (A.26) is given in [8] (see formula (3.14) of the paper). The derivation is based on a two step procedure of minimization of (2.3): the first step is the minimization over all unit measures with a given charge $\beta_1 \in (0, 1)$ of the "band" $[a_2, b_2]$ of the support, and the second is the minimization of this minimum over β_1 .

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