Steepest descent analysis of the Riemann-Hilbert problem for multiple Hermite polynomials

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Orthogonal Polynomials

Major developments in the field of orthogonal polynomials in last decade:

- **Barry Simon (and collaborators):**
    - **OPUC, OPRL, CMV matrix, Verblunsky coefficients, . . .**
- **Percy Deift (and collaborators):**
    - **Riemann-Hilbert problem, Steepest descent, Random matrices, . . .**

RH problem for Orthogonal Polynomials

- $P_n(x) = x^n + \cdots$ **is monic orthogonal polynomial with weight w on $\mathbb{R}$ if**

$$\int_{-\infty}^{\infty} P_n(x) x^k w(x) dx = 0 \qquad \text{for } k = 0, 1, \ldots, n-1.$$

- **Orthogonal polynomials are characterized by a 2×2-matrix valued Riemann-Hilbert problem** **Fokas, Its, Kitaev (1992)**
  - $Y : \mathbb{C} \setminus \mathbb{R} \to \mathbb{C}^{2\times 2}$ **is analytic such that**

$$Y_+(x) = Y_-(x) \begin{pmatrix} 1 & w(x) \\ 0 & 1 \end{pmatrix} \qquad \text{for } x \in \mathbb{R},$$

$$Y(z) \begin{pmatrix} z^{-n} & 0 \\ 0 & z^n \end{pmatrix} \to I \qquad \text{as } z \to \infty \qquad \textbf{(not normalized)}$$

Solution of RH problem

$$Y(z) = \begin{pmatrix} P_n(z) & \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{P_n(s)w(s)}{s-z} ds \\ c_{n-1}P_{n-1}(z) & \frac{c_{n-1}}{2\pi i}\int_{-\infty}^{\infty} \frac{P_{n-1}(s)w(s)}{s-z} ds \end{pmatrix}$$

Asymptotic analysis

- Deift/Zhou steepest descent analysis for Riemann-Hilbert problems can be applied to the Riemann-Hilbert problem for orthogonal polynomials

  Deift, Kriecherbauer, McLaughlin, Venakides, Zhou (1999)

- Asymptotics for orthogonal polynomials, recurrence coefficients, . . .
- Universality proofs in random matrix theory, . . .

Multiple orthogonal polynomials of type II

- **Given m weights $w_1, \ldots, w_m$ and multi-index $\vec{n} = (n_1, \ldots, n_m)$.**
- **$P_{\vec{n}}$ is a polynomial such that**

$$\left.\begin{array}{l} \displaystyle\int_{-\infty}^{\infty} P_{\vec{n}}(x) x^k w_j(x) dx = 0, \\ \qquad \text{for } k = 0, \ldots, n_j - 1, \quad j = 1, \ldots, m \end{array}\right\} \quad \text{type II orthogonality}$$

$$P_{\vec{n}} \text{ is monic of degree } |\vec{n}| = n_1 + \cdots + n_m \qquad \text{normalization}$$

- **Need not always exist, and if it exists need not be unique.**

RH problem for MOP of type II (case $m = 2$)

- **MOP Riemann-Hilbert problem** Van Assche, Geronimo, AK (2001)
  - $Y : \mathbb{C} \setminus \mathbb{R} \to \mathbb{C}^{3\times 3}$ **is analytic**
  - $Y_+(x) = Y_-(x) \begin{pmatrix} 1 & w_1(x) & w_2(x) \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ **for** $x \in \mathbb{R}$
  - $Y(z) \begin{pmatrix} z^{-n_1-n_2} & 0 & 0 \\ 0 & z^{n_1} & 0 \\ 0 & 0 & z^{n_2} \end{pmatrix} \to I_3$ **as** $z \to \infty$

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Solution

- **Solution contains multiple orthogonal polynomials**

$$Y = \begin{pmatrix} P_{n_1,n_2} & C(P_{n_1,n_2}w_1) & C(P_{n_1,n_2}w_2) \\ c_1 P_{n_1-1,n_2} & c_1 C(P_{n_1-1,n_2}w_1) & c_1 C(P_{n_1-1,n_2}w_2) \\ c_2 P_{n_1,n_2-1} & c_2 C(P_{n_1,n_2-1}w_1) & c_2 C(P_{n_1,n_2-1}w_2) \end{pmatrix}$$

where

$$c_1 = -2\pi i \left[\int P_{n_1-1,n_2}(s) s^{n_1} w_1(s) ds \right]^{-1}$$

$$c_2 = -2\pi i \left[\int P_{n_1,n_2-1}(s) s^{n_2} w_2(s) ds \right]^{-1}$$

and C is the Cauchy transform $$C(f)(z) = \frac{1}{2\pi i} \int \frac{f(s)}{s-z} ds$$

- **RH problem is solvable if and only if the MOP P_{n_1,n_2} exists and is unique.**

Multiple Orthogonal Polynomials of type I

- **Given m weights $w_1, \dots, w_m$ and multi-index $\vec{n} = (n_1, \dots, n_m)$.**
- **Let $A_{\vec{n}}^{(j)}$, $j = 1, \dots, m$, be polynomials with $\deg A_{\vec{n}}^{(j)} \leq n_j - 1$, so that**

$$Q_{\vec{n}}(x) = \sum_{j=1}^{m} A_{\vec{n}}^{(j)}(x) w_j(x)$$

satisfies

$$\int_{-\infty}^{\infty} x^k Q_{\vec{n}}(x) dx = \begin{cases} 0, & k = 0, 1, \dots, |\vec{n}| - 2, \qquad \textbf{type I orthogonality} \\ 1, & k = |\vec{n}| - 1, \qquad \textbf{normalization} \end{cases}$$

RH problem for MOP of type I (case $m = 2$)

- **MOP Riemann-Hilbert problem for type I** Van Assche, Geronimo, AK (2001)
- $X : \mathbb{C} \setminus \mathbb{R} \to \mathbb{C}^{3\times 3}$ is analytic
- $$X_+ = X_- \begin{pmatrix} 1 & 0 & 0 \\ -w_1 & 1 & 0 \\ -w_2 & 0 & 1 \end{pmatrix}$$ on $\mathbb{R}$
- $$X(z) \begin{pmatrix} z^{n_1+n_2} & 0 & 0 \\ 0 & z^{-n_1} & 0 \\ 0 & 0 & z^{-n_2} \end{pmatrix} \to I_3$$
- Comparing the RH problems we easily see that

$$X = Y^{-t}$$ (inverse transpose)

Solution

- **Solution contains multiple orthogonal polynomials of type I**

$$X = 2\pi i \begin{pmatrix} C(Q_{n_1,n_2}) & A^{(1)}_{n_1,n_2} & A^{(2)}_{n_1,n_2} \\ k_1 C(Q_{n_1+1,n_2}) & k_1 A^{(1)}_{n_1+1,n_2} & k_1 A^{(2)}_{n_1+1,n_2} \\ k_2 C(Q_{n_1,n_2+1}) & k_2 A^{(1)}_{n_1,n_2+1} & k_2 A^{(2)}_{n_1,n_2+1} \end{pmatrix}$$

where

$$k_1 = (2\pi i)^{-1} \int P_{n_1,n_2}(s) s^{n_1+1} w_1(s)\,ds$$

$$k_2 = (2\pi i)^{-1} \int P_{n_1,n_2}(s) s^{n_2+1} w_2(s)\,ds$$

Multiple Hermite polynomials

- Multiple Hermite polynomials are associated with weights

$$w_j(x) = e^{-n(\frac{1}{2}x^2 - a_j x)}$$

  where $a_j \neq a_k$ if $j \neq k$.

  - Multiple Hermite polynomials exist and are unique; both type I and II.
  - There exist integral representations, differential equation, Rodrigues formula, explicit recurrences, ...

- More generally, if weights are

$$w_j(x) = e^{-n(V(x) - a_j x)}$$

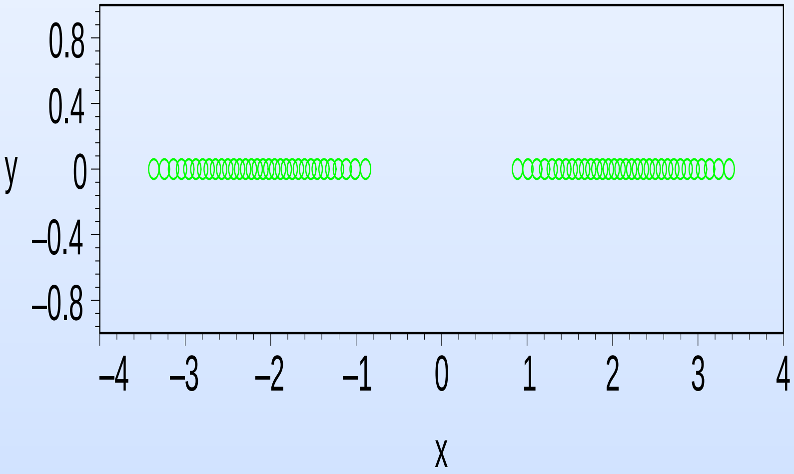
  where $a_j \neq a_k$ if $j \neq k$, then MOP exist and are unique.

Zeros of multiple Hermite polynomials

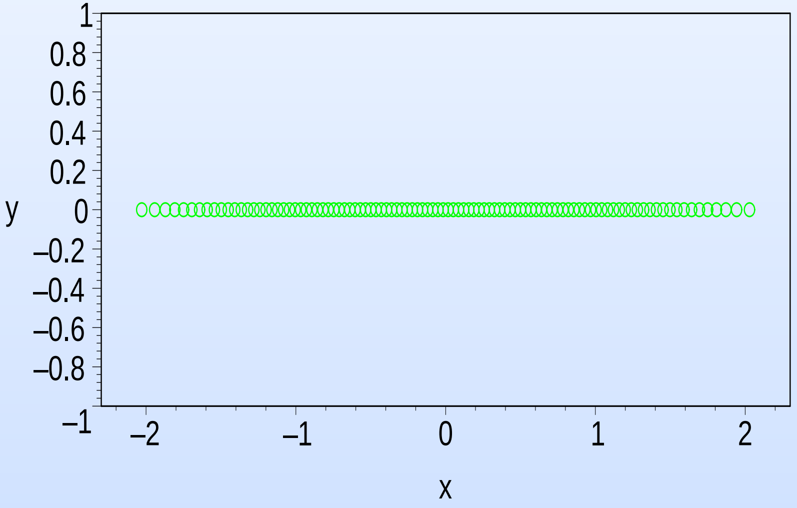
- **Type II orthogonality** $a_1 = -a_2 = a$, $n_1 = n_2 = n/2$, $P_n = P_{n_1,n_2}$

$$\int_{-\infty}^{\infty} P_n(x) x^k e^{-n(\frac{1}{2}x^2 \pm ax)} dx = 0, \qquad k = 0, \ldots, \frac{n}{2} - 1.$$

- **Zeros are real and simple**



$a = 2$, $n = 60$,



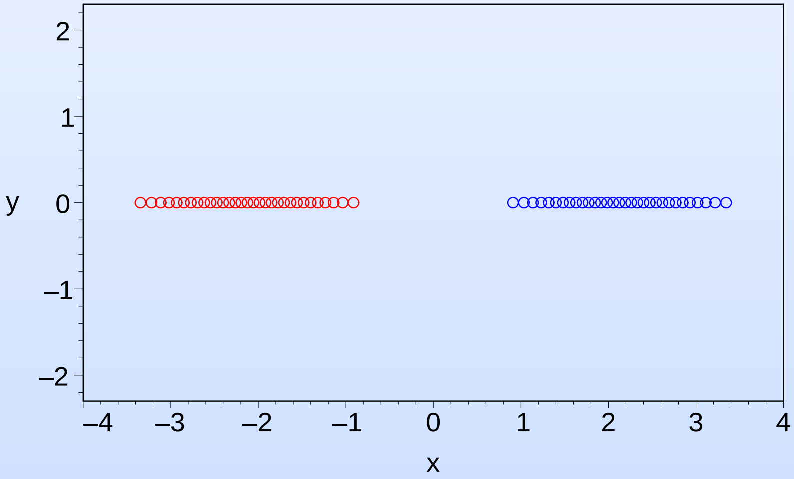
$a = 0.4$, $n = 100$

Zeros of multiple Hermite polynomials

- **Type I orthogonality** $a_1 = -a_2 = a,\, n_1 = n_2 = n/2,\, A_n^{(j)} = A_{n_1,n_2}^{(j)}$

$$\int_{-\infty}^{\infty} x^k \left(A_n^{(1)}(x) e^{-n(\frac{1}{2}x^2 - ax)} + A_n^{(2)}(x) e^{-n(\frac{1}{2}x^2 + ax)} \right) dx = 0, \qquad k = 0, \ldots$$

- **Zeros of $A_n^{(1)}$ and $A_n^{(2)}$ are not always real**



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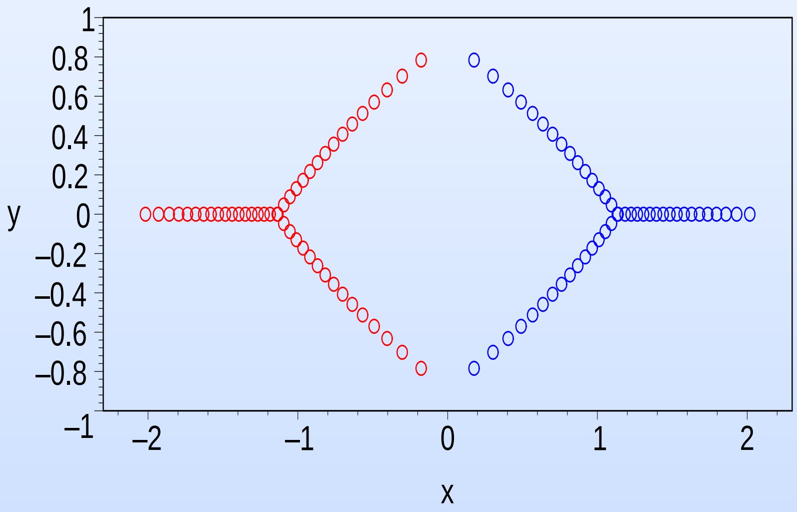


$a = 0.4,\ n = 100$

Steepest descent analysis

- **We applied Deift/Zhou steepest descent analysis to the RH problem for multiple orthogonal polynomials in three cases**
  - **Case $a > 1$: zeros of P_n are on two intervals** **Bleher, AK (2004)**
  - **Case $0 < a < 1$: zeros of P_n are on one interval; some zeros of $A_n^{(j)}$ are complex** **Aptekarev, Bleher, AK (2005)**
  - **Case $a = 1$** **Bleher, AK (arxiv 2006)**

Random matrices with external source

- Motivation to study asymptotics of multiple Hermite polynomials comes from random matrix model with external source.
- Probability measure on $n \times n$ Hermitian matrices **Brézin, Hikami (1996-98)**

$$\frac{1}{Z_n} e^{-\operatorname{Tr}(V(M)-AM)} dM$$

  - Model depends on eigenvalues of A. Suppose distinct eigenvalues are $a_1, \dots, a_m$ with multiplicities $n_1, \dots, n_m$.
  - Average characteristic polynomial $\mathbb{E}\left[z - M\right]$ is a MOP of type II with respect to weights

$$w_j(x) = e^{-(V(x)-a_j x)}, \qquad j = 1, \dots, m$$

and multi-index $\vec{n} = (n_1, \dots, n_m)$.

MOP correlation kernel

- Probability measure on $n \times n$ Hermitian matrices

$$\frac{1}{Z_n} e^{-\operatorname{Tr}(V(M)-AM)} dM$$

  - Eigenvalue correlations are determinantal with kernel $K_{\vec{n}}$ built out of MOP of type I and II.
  - By a Christoffel-Darboux formula for MOP we can give the correlation kernel in terms of the solution of the RH problem for MOP of type II

$$K_{\vec{n}}(x,y) = \frac{1}{2\pi i(x-y)} \begin{pmatrix} 0 & w_1(y) & w_2(y) \end{pmatrix} Y^{-1}(y) Y(x) \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

for $m = 2$ **Bleher, AK (2004)**

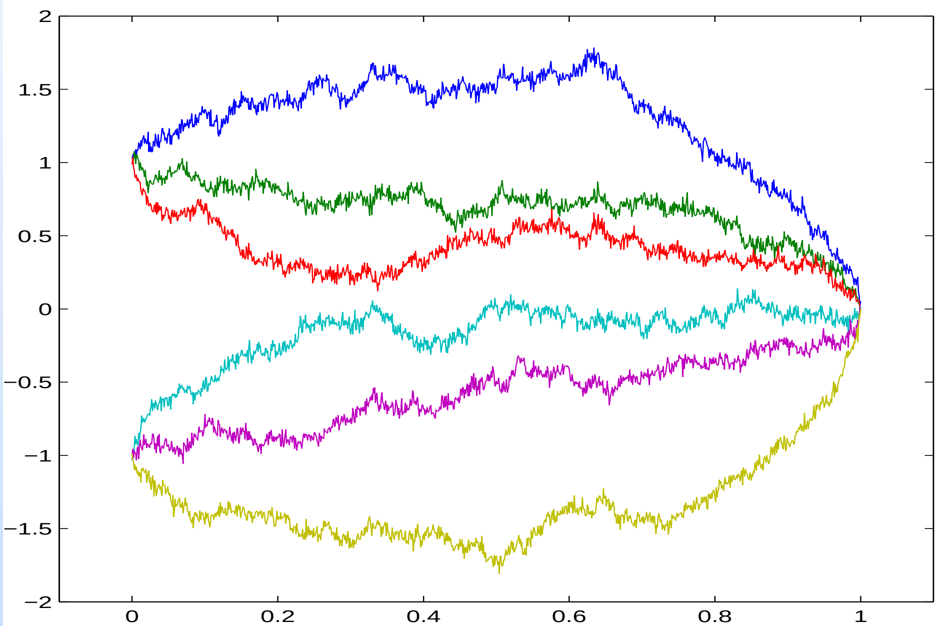
for general m **Daems, AK (2004)**

Gaussian model with external source

- **The eigenvalues of Gaussian random matrix model with external source**

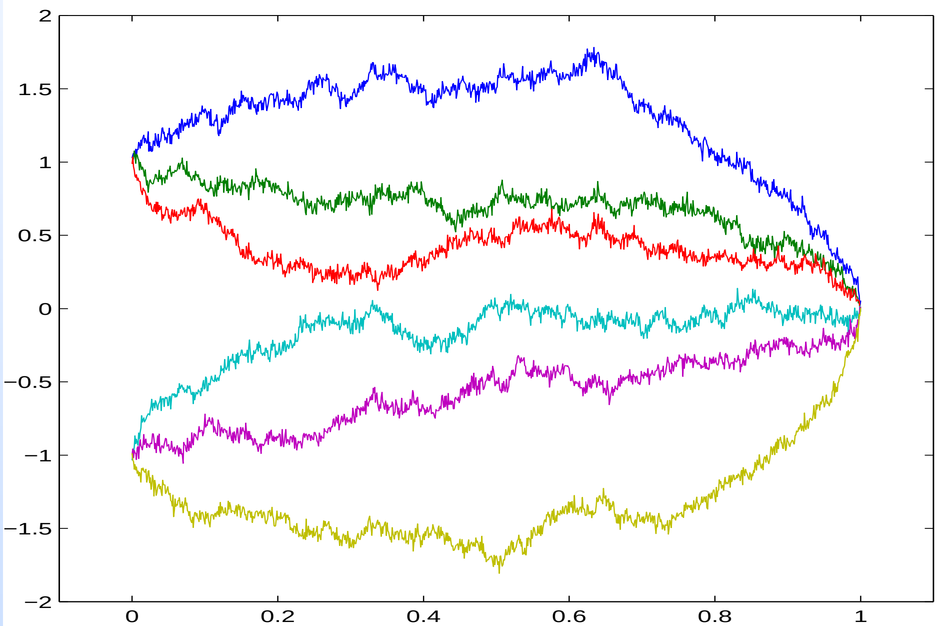
$$\frac{1}{Z_n} e^{-\operatorname{Tr}(\frac{1}{2}M^2 - AM)} dM$$

have equivalent interpretation in terms of non-intersecting Brownian paths.



Non-intersecting Brownian paths

- Consider n Brownian motions that **Karlin, McGregor (1959)**
  - start at time $t = 0$ at m points $b_1, \ldots, b_m$, where n_j paths start of b_j.
  - end at time $t = 1$ at the same point
  - are conditioned not to intersect at intermediate times.

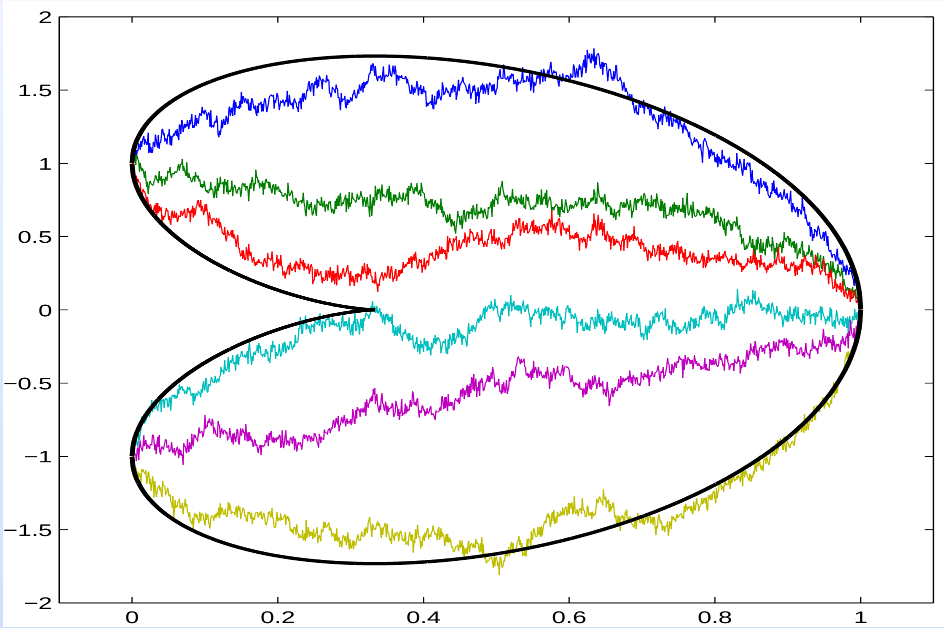


- At any intermediate time t the positions of the paths are distributed as the eigenvalues of the Gaussian matrix model with external source with multiplicities $n_1, \ldots, n_m$ and values a_j's that depend on t.

Large n limit of Brownian paths

Let $n \to \infty$ while decreasing the variance of the Brownian paths to $1/n$

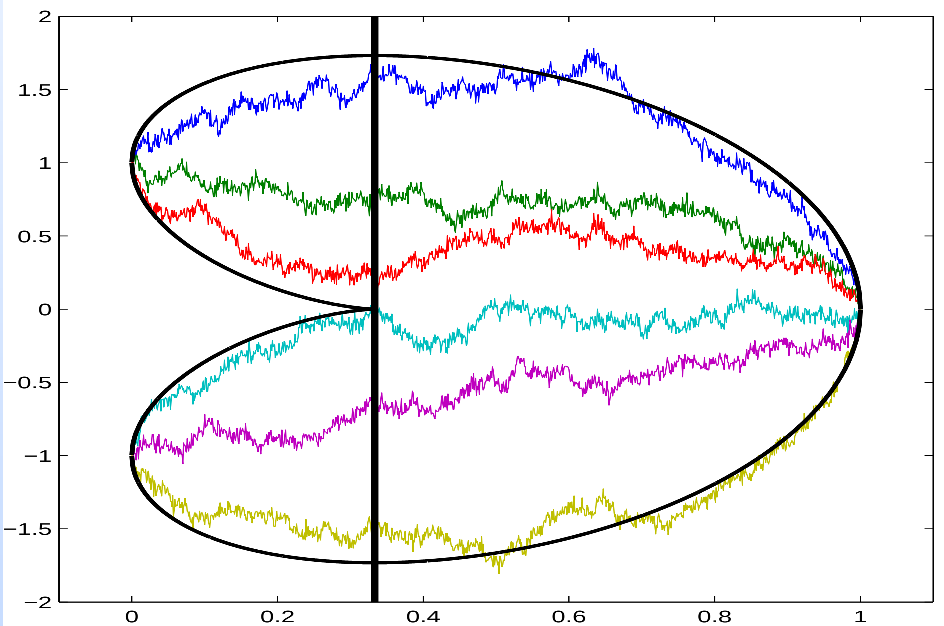
- **As $n \to \infty$, the paths fill out a region.**



Large n limit of Brownian paths

Let $n \to \infty$ while decreasing the variance of the Brownian paths to $1/n$

- **As $n \to \infty$, the paths fill out a region.**
  - **Two groups of paths are separated until critical time.**
  - **Afterwards they continue as one group.**
  - **Cusp singularity at critical time.**



Large n limit of Gaussian model: Global regime

$$\frac{1}{Z_n} e^{-n \operatorname{Tr}(\frac{1}{2}M^2 - AM)} dM$$

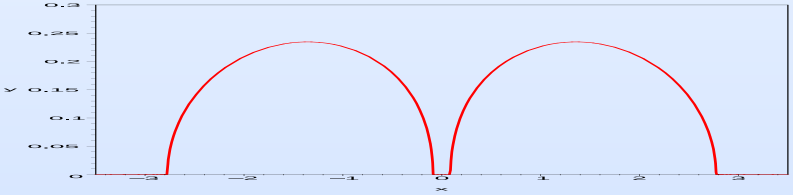
where A has two eigenvalues $\pm a$ of equal multiplicity.

- **The limiting mean eigenvalue density exists** **Pastur (1972)**

$$\rho(x) = \lim_{n\to\infty} \frac{1}{n} K_{n/2,n/2}(x,x)$$

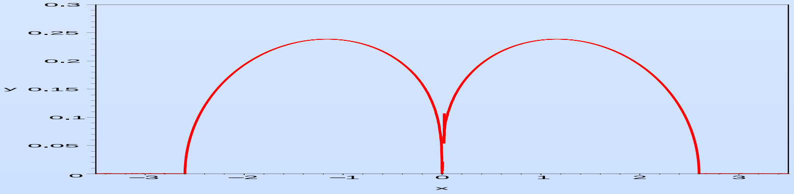
- $a > 1$

  two intervals



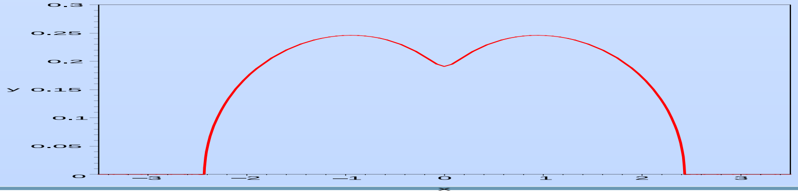
- $a = 1$

  critical case



- $a < 1$

  one interval



Local eigenvalue regime

- Rescale eigenvalues in random matrix model $\frac{1}{Z_n} e^{-n \operatorname{Tr} V(M)} dM$ around some reference point x_0: $\lambda \mapsto cn^\gamma (\lambda - x_0)$
- Corresponds to rescaling of correlation kernel

$$\frac{1}{cn^\gamma} K_{\vec{n}}(x_0 + \frac{x}{cn^\gamma}, x_0 + \frac{y}{cn^\gamma})$$

- Scaling limits of the correlation kernel are universal
  - Sine kernel in the bulk $\frac{\sin \pi (x-y)}{\pi (x-y)}$
  - Airy kernel at the edge $\frac{\operatorname{Ai}(x) \operatorname{Ai}'(y) - \operatorname{Ai}'(x) \operatorname{Ai}(y)}{x - y}$

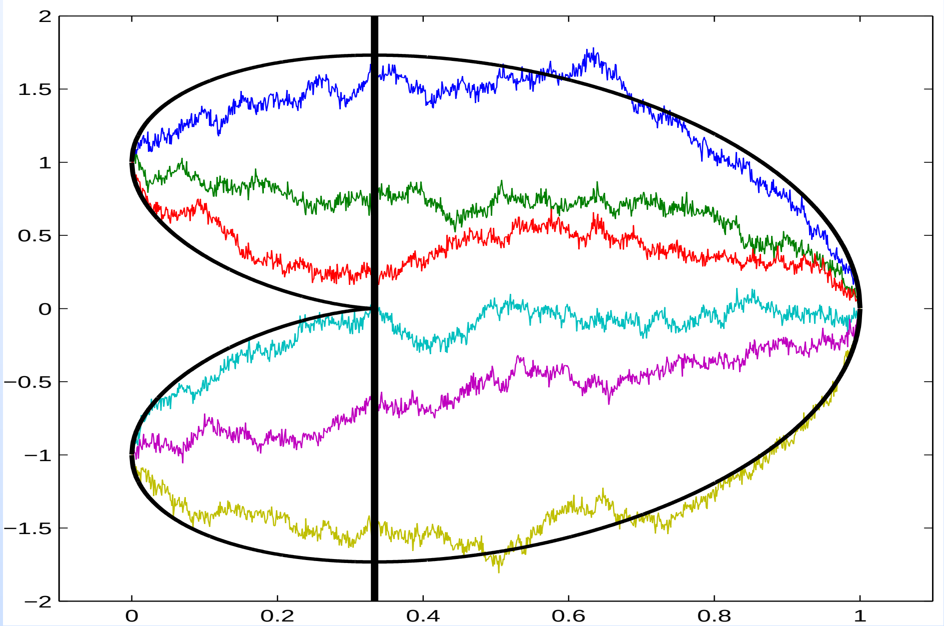
Pastur, Shcherbina (1997), for quartic V Bleher, Its (1999)

for real analytic V Deift, Kriecherbauer, McLaughlin, Venakides, Zhou (1999)

Local eigenvalue regime

- **Sine kernel and Airy kernel also appear as limits in the Gaussian random matrix model with external source.**
- **New kernels appears at the cusp singularity: Pearcey kernels**

Brézin, Hikami (1997-98), Tracy, Widom (2006)



Pearcey kernel

- **New limiting kernels appear at $x^* = 0$ in the double scaling limit $n \to \infty$, $a \to 1$ such that $b = 2n^{1/2}(a-1)$ is constant**

$$\boxed{\frac{p(x)q''(y) - p'(x)q'(y) + p''(x)q(y) - bp(x)q(y)}{x-y}}$$

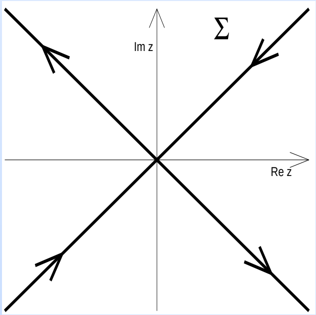
where p and q are Pearcey integrals

$$p(x) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{-\frac{1}{4}s^4 - \frac{b}{2}s^2 + isx}\,ds, \qquad q(y) = \frac{1}{2\pi}\int_{\Sigma} e^{\frac{1}{4}t^4 + \frac{b}{2}t^2 + ity}\,dt.$$

- **p and q satisfy third order ODEs**

$$p''' = xp + bp'$$

$$q''' = -yq + bq'$$



New features in RH problem steepest descent analysis

- **Our approach is based on the RH problem for multiple Hermite polynomials and the Deift/Zhou steepest descent analysis.**
  - **Normalization of the RH problem needs two measures that satisfy certain equilibrium conditions. In the Gaussian case there is an explicit description in terms of algebraic functions on a Riemann surface.**
  - **For $a < 1$, two complex branch points are not on the main sheet. This requires additional global opening of lenses.**
  - **For $a \approx 1$ (double scaling limit), we need a construction of a RH problem for Pearcey integrals and relate it with the RH problem for multiple Hermite polynomials. To achieve good matching condition we use a modified Riemann surface, which can also be interpreted as modified equilibrium measures.**

Happy birthday!!

Happy Birthday, Percy !