On the asymptotic analysis of the distribution functions of the random matrix theory.

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The Problem

Let $K_{airy}^{(s)}$ be the trace-class operator with kernel

$$K_{airy}(u,v) = \frac{\mathrm{Ai}\,(u)\mathrm{Ai}\,'(v) - \mathrm{Ai}\,(v)\mathrm{Ai}\,'(u)}{u-v}$$

acting on $L^2(-s,\infty)$, $s>0$. Here $\mathrm{Ai}\,(x)$ is the Airy function.

We are interested in the behavior of

$$\det\left(I - K_{airy}^{(s)}\right) \equiv F_{TW}(-s),$$

as $s \to \infty$.

We prove

Theorem 1.

The large-s asymptotics of the Fredholm determinant $\det\left(I - K_{airy}^{(s)}\right)$ *are given by the formula*

$$\det\left(I - K_{airy}^{(s)}\right) = -\frac{s^3}{12} - \frac{1}{8}\ln s + \kappa$$

$$+O(s^{-3/2}),$$

where (The Tracy-Widom conjecture)

$$\kappa = \frac{1}{24}\ln 2 + \zeta'(-1),$$

and $\zeta(s)$ is the Riemann zeta-function.

(**P. Deift**, **I. Krasovsky**, **A. I.**)

Remark 1. (The difficulty)

The standard procedure:

- $\det\left(I - K^{(s)}\right) \to \det\left(I - \gamma K^{(s)}\right)$

- $I + R_\gamma^{(s)} = \left(I - K^{(s)}\right)^{-1}, \quad R_\gamma^{(s)} \sim ..., \quad s \to \infty$

- $\frac{d}{d\gamma} \ln \det\left(I - \gamma K^{(s)}\right) = -\mathrm{trace}\, R_\gamma^{(s)}$

Needed: the uniform asymptotics of $R_\gamma^{(s)}$ for all $0 \leq \gamma \leq 1$.

The reality: can be achieved for $0 \leq \gamma \leq \gamma_0$ with $\gamma_0 < 1$ but not for $\gamma_0 = 1$.

One of the ways to see this:

Observe that

$$\det\left(I - \gamma K_{airy}^{(s)}\right) \equiv F_{TW}^{\gamma}(-s)$$

$$= \exp\left\{-\int_{-s}^{\infty}(s+x)u^2(x)dx\right\},$$

where

$$u_{xx} = xu + 2u^3, \quad u(x) \sim \gamma \mathrm{Ai}\,(x), \quad x \to \infty.$$

The behavior of the solution $u(x)$ at $x \to -\infty$:

- $0 \le \gamma < 1$ (Ablowitz-Segur solution):

$$u(x) = (-x)^{-1/4}\alpha \sin\left\{\frac{2}{3}(-x)^{3/2} - \frac{3}{4}\ln(-x) + \phi\right\}$$

$$+o(x^{-1/4}), \quad x \to -\infty.$$

$$\alpha^2 = -\frac{1}{\pi}\ln(1-\gamma^2),$$

$$\phi = -\frac{3}{2}\alpha^2 \ln 2 - \frac{\pi}{4} + \arg\Gamma\left(i\frac{\alpha^2}{2}\right)$$

- $\gamma = 1$ (Hastings-McLeod solution):

$$u(x) \sim \sqrt{-\frac{x}{2}}, \quad x \to -\infty$$

- $0 \leq \gamma < 1$:

$$\ln \det \left(I - \gamma K_{airy}^{(s)} \right) \sim -\frac{2\alpha^2}{3} s^{3/2}, \quad s \to \infty$$

- $\gamma = 1$:

$$\ln \det \left(I - K_{airy}^{(s)} \right) \sim -\frac{1}{12} s^3, \quad s \to \infty$$

To summarize:

The core of the difficulty is the necessity to integrate the asymptotics of

$$\frac{d}{d\gamma} \ln \det \left(I - K_{airy}^{(s)} \right) \equiv -\mathrm{trace}\, R_{\gamma}^{(s)},$$

over a **boundary layer** as $\gamma - \to 1$.

Airy as a limit of Laguerre

Let $K_n^{(\alpha)}$ be the trace-class operator with kernel

$$K_n(x,y) = \frac{1}{4}\frac{\omega_n(x)\omega_{n-1}(y) - \omega_n(y)\omega_{n-1}(x)}{y-x}$$

acting on $L^2(\alpha,\infty)$, $\alpha > 0$. Here

$$\omega_k(x) = e^{-2nx}p_k(x), \qquad k = 0, 1, \ldots,$$

and $p_k(x) \equiv p_k(x;n)$ denote the scaled Laguerre polynomials,

$$p_k(x) = 2\sqrt{n}L_k^{(0)}(4nx).$$

We note that

$$\int_0^\infty e^{-4nx} p_k(x) p_m(x) dx = \delta_{k,m}, \qquad k, m = 0, 1 \ldots,$$

and

$$p_k(x) = c_k x^k + \cdots$$

$$c_k = (-1)^k \frac{2\sqrt{n}}{k!} (4n)^k$$

Consider the Fredholm determinant:

$$D_n(\alpha) = \det\left(I - K_n^{(\alpha)}\right).$$

Proposition 1.

$$\lim_{n\to\infty} D_n\left(1 - \frac{s}{(2n)^{2/3}}\right) = \det\left(I - K_{airy}^{(s)}\right)$$

(cf. **Forrester, Tracy, Widom, Deift, Goev, Vanlessen**)

An outline of the proof of Theorem 1

Proposition 1. (The Riemann-Hilbert analysis)

$$\frac{d}{d\alpha} \ln D_n(\alpha) = \frac{n^2}{\alpha}(1-\alpha)^2 + \frac{\alpha}{4(1-\alpha^2)}$$

$$+O\left(\frac{1}{n(1-\alpha)^{5/2}}\right),$$

$$\forall \alpha \in \left[0, 1 - \frac{s_0}{(2n)^{2/3}}\right], \quad n > \frac{s_0^{3/2}}{2}.$$

Conclusion: The theorem 1 will be proven if the behavior of $D_n(\alpha)$ at $\alpha \to 0$ with n fixed is known

Proposition 2. (Multiple integral representation)

$$D_n(\alpha) = \frac{1}{C_n n!} \int_0^\alpha \cdots \int_0^\alpha \prod_{0 \le j < k \le n-1} (x_j - x_k)^2$$

$$\times \prod_{j=0}^{n-1} e^{-4nx_j} dx_0 \ldots dx_{n-1},$$

$$C_n = \frac{1}{n!} \int_0^\infty \cdots \int_0^\infty \prod_{0 \le j < k \le n-1} (x_j - x_k)^2$$

$$\times \prod_{j=0}^{n-1} e^{-4nx_j} dx_0 \ldots dx_{n-1}$$

Observation 1.

$$D_n(\alpha) = \frac{1}{C_n}\left(\frac{\alpha}{2}\right)^2 A_n \left(1 + O_n(\alpha)\right),$$

$$\alpha \to 0$$

$$A_n = \frac{1}{n!}\int_{-1}^{1} \cdots \int_{-1}^{1} \prod_{0\le j<k\le n-1} (t_j - t_k)^2$$

$$\times dt_0 \ldots dt_{n-1},$$

$$\left(\quad x_j = \frac{\alpha}{2}(t_j + 1) \quad \right)$$

Observation 2. (Selberg Formulae)

$$C_n = \prod_{k=0}^{n-1} c_k^{-2} = (4n)^{-n^2} \prod_{k=0}^{n-1} (k!)^2.$$

(Laguerre polynomials)

$$A_n = \prod_{k=0}^{n-1} d_k^{-2} = \prod_{k=0}^{n-1} \frac{2^{2k}(k!)^4}{[(2k)!]^2} \frac{2}{2k+1}.$$

(Legendre polynomials)

Corollary 1.

$$\ln D_n(\alpha) = \left(\frac{3}{2} + \ln\alpha\right) n^2 - \frac{1}{12}\ln\frac{n}{2}$$

$$+\zeta'(-1) + \delta_n + O_n(\alpha),$$

$$\delta_n \to 0, \quad n \to \infty$$

Corollary 2. The theorem 1

The Widom-Dyson constant. The Sine kernel case

$$D_n(\alpha) := \det T_n[\phi],$$

where

$$T_n[\phi] := \{\phi_{j-k}\}, \quad k = 0, ..., n-1,$$

and

$$\phi_k = \int_C \phi(z) z^{-k-1} \frac{dz}{2\pi i},$$

$$\phi(z) = \chi_{C_\alpha}(z),$$

$$C_\alpha : \alpha \le \arg z \le 2\pi - \alpha.$$

Observation

$$\lim_{n\to\infty} D_n|_{\alpha=2s/n} = \det(1-K_{\text{sine}}),$$

where

$$K_{\text{sine}} : L_2(0,2s) \to L_2(0,2s),$$

$$K_{\text{sine}}(x,y) = \frac{\sin(x-y)}{\pi(x-y)}.$$

Dyson's conjecture

$$\det(1-K_{\text{sine}}) \sim e^{-\frac{s^2}{2}} s^{-\frac{1}{4}}$$

$$\times e^{3\zeta'(-1)+\frac{1}{12}\ln 2}, \qquad s\to\infty. \tag{1}$$

(proven by **Krasovsly** and **Ehrhardt**)

The third proof

(**Deift**, **Krasovsky**, **Zhou**, **I**)

The RH problem to be solved is the following one.

- $Y(z)$ is holomorphic for all $z \notin \Gamma_\alpha$

- $Y(\infty) = I$

- $Y_-(z) = Y_+(z) \begin{pmatrix} 2 & -z^n \\ z^{-n} & 0 \end{pmatrix}, \quad z \in \Gamma_\alpha$

$$\Gamma_\alpha = \{z : -\alpha \le \arg z \le \alpha\}.$$

$$\frac{d^2}{d\alpha^2} \ln D_n(\alpha) = -\frac{n^2}{\sin^2 \alpha}[Y_{12}(0)]^2.$$

$$\begin{pmatrix} 2 & -z^n \\ z^{-n} & 0 \end{pmatrix} \neq \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \Lambda \begin{pmatrix} 1 & 0 \\ * & 1 \end{pmatrix}$$

g - **function:**

$$g(z) = \frac{z + 1 + \sqrt{(z - e^{i\alpha})(z - e^{-i\alpha})}}{2z}$$

- $g(z)$ is holomorphic for all $z \notin \Gamma_\alpha$
- $g(\infty) = 1$
- $\left|\frac{g_+(z)}{g_-(z)}\right| < 1, \quad z \neq e^{\pm i\alpha}$
- $g_+(z)g_-(z) = \frac{\kappa}{z}, \quad \kappa = \cos^2 \frac{\alpha}{2}$

Put

$$X(z) = \kappa^{-\frac{n}{2}\sigma_3} Y(z) (g(z))^{-n\sigma_3} \kappa^{\frac{n}{2}\sigma_3}$$

$$\sigma_3 = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$$

- $X(z)$ is holomorphic for all $z \notin \Gamma_\alpha$
- $X(\infty) = I$
- $X_-(z) = X_+(z) \begin{pmatrix} 2\left(\frac{g_+(z)}{g_-(z)}\right)^n & -1 \\ 1 & 0 \end{pmatrix}$

  $z \in \Gamma_\alpha$

Since $\left|\frac{g_+(z)}{g_-(z)}\right| < 1, \quad z \neq e^{\pm i\alpha}$, we expect that

$$X(z) \sim X^0(z) :$$

- $X^0(z)$ is holomorphic for all $z \notin \Gamma_\alpha$

- $X^0(\infty) = I$

- $X^0_-(z) = X^0_+(z) \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

  $z \in \Gamma_\alpha$

SOLUTION to the MODEL PROBLEM:

$$X^0(z) = \begin{pmatrix} \frac{\delta+\delta^{-1}}{2} & \frac{\delta-\delta^{-1}}{2i} \\ -\frac{\delta-\delta^{-1}}{2i} & \frac{\delta+\delta^{-1}}{2} \end{pmatrix},$$

where

$$\delta(z) = \left(\frac{z - e^{i\alpha}}{z + e^{-i\alpha}}\right)^{\frac{1}{4}}.$$

The leading term of Widom's asymptotics, i.e.

$$\frac{d^2}{d\alpha^2} \ln D_n(\alpha) \sim -\frac{n^2}{4} \sec^2 \frac{\alpha}{2} , \qquad (2)$$

(formally) follows. (**Deift**, **Zhou**, **I**; 1997)

• More carefull analysis (see Appendix) along the lines indicated allows to obtain the following (rigorous) extension of (2).

$$\frac{d^2}{d\alpha^2} \ln D_n(\alpha) = -\frac{n^2}{\sin^2 \alpha} \Delta(n, \alpha), \tag{3}$$

$$\Delta(n, \alpha) = \sin^2 \frac{\alpha}{2} - \frac{\cos^2(\alpha/2)}{4n^2}$$

$$+ O\left(\frac{1}{n^3 \sin^3(\alpha/2)}\right) \sin^2 \alpha,$$

$$n \to \infty, \quad \frac{2s}{n} \le \alpha \le \pi, \quad s \ge s_0.$$

• From the multiple integral representation of $D_n(\alpha)$ one obtains

$$\ln D_n(\alpha) = n^2 \ln(\pi - \alpha) - n \ln 2\pi$$

$$+ \ln A_n + O((\pi - \alpha)^2), \qquad (4)$$

where

$$A_n = \prod_{k=0}^{n-1} \frac{2^{2k}(k!)^4}{[(2k)!]^2} \frac{2}{2k+1}$$

$$= e^{c_0} n^{-1/4} (2\pi)^n 2^{-n^2} (1 + o(1)), \quad n \to \infty.$$

(3) and (4) imply (1)

(**Deift**, **Krasovsky**, **Zhou**, **I**)

Appendix

Put

$$\Phi(\lambda) := X^{-1}(-1)X(z(\lambda)),$$

where

$$z(\lambda) = \frac{1 + i\lambda \tan\frac{\alpha}{2}}{1 - i\lambda \tan\frac{\alpha}{2}}$$

maps the interval $[-1, 1]$ to the arc Γ_α.

- $\Phi(\lambda)$ is holomorphic for all $\lambda \notin [-1, 1]$

- $\Phi(\infty) = I$

- $$\Phi_-(\lambda) = \Phi_+(\lambda) \begin{pmatrix} 2\left[\frac{1-\sqrt{1-\lambda^2}\sin\frac{\alpha}{2}}{1+\sqrt{1-\lambda^2}\sin\frac{\alpha}{2}}\right]^n & -1 \\ 1 & 0 \end{pmatrix},$$

$$\lambda \in (-1, 1)$$

Note:

- The Φ - problem is regular in the neighborhood of $\alpha = \pi$

- The following relation takes place

$$\frac{d^2}{d\alpha^2} \ln D_n(\alpha) = -\frac{n^2}{\sin^2\alpha}\Delta(n,\alpha),$$

where

$$\Delta(n,\alpha) = \left[\Phi^{-1}(-i\cot\frac{\alpha}{2}; n,\alpha)\Phi(i\cot\frac{\alpha}{2}; n,\alpha)\right]^2_{12}.$$

• D_n satisfies Painlevé VI equation:

$$\eta(t) \equiv t(t-1)\frac{d}{dt}\ln D_n, \quad t \equiv e^{-2i\alpha}$$

$$\left(\frac{d\eta}{dt}\right)^4 = \left(\frac{d\eta}{dt} - \frac{n^2}{4}\right)\left(t(t-1)\frac{d^2\eta}{dt^2}\right)^2$$

$$+\left[2\left(\frac{d\eta}{dt} - \frac{n^2}{4}\right)\left(t\frac{d\eta}{dt} - \eta\right) - \left(\frac{d\eta}{dt}\right)^2 + \frac{n^2}{2}\frac{d\eta}{dt}\right]^2.$$

$$\Delta = \frac{1-t}{n^2}\,\frac{d\eta}{dt} + \frac{1}{n^2}\eta.$$

(**Deift**, **Zhou**, **I**; **Tracy**, **Widom**)