Integrable Systems, Random Matrices and Applications: Conference in honor of Percy Deift's 60th birthday

---

Courant Institute of Mathematical Sciences
New York University
May 22–26, 2006

HAPPY BIRTHDAY, PERCY!

Entanglement entropy of fermions in any dimension and the Widom conjecture

---

Dimitri Gioev, University of Rochester
(joint work with Israel Klich, Caltech)

Supported by the National Science Foundation Grant DMS–0556049

Outline of the talk

1. Definition of the entanglement entropy (EE)
2. Known results
    - $D = 1$: spin chain systems
    - $D > 1$: Area Law for EE: harmonic lattice, Kitaev model
3. New results
    - $D \geq 1$: violation of the Area Law for free fermions in lattice and continuous systems: two-sided estimates, explicit formula for EE, fractal boundaries
    - connection with the Widom conjecture, strong Szegö limit theorem and Onsager's work on the $2D$ Ising model

Definition of EE in the simplest case

Consider two finite dimensional Hilbert spaces H_A, H_B.

EE of a state $\psi \in H_A \otimes H_B$ measures how far it is from a product state $\phi_A \otimes \phi_B$. More precisely, recall that any state ψ can be written as

$$\psi = \sum_i c_i \phi_{A,i} \otimes \phi_{B,i}, \qquad 0 \leq c_i \leq 1,$$

where $(\phi_{A,i}, \phi_{A,j}) = (\phi_{B,i}, \phi_{B,j}) = \delta_{i,j}$ (the Schmidt decomposition).

The EE of the state ψ is defined by

$$S = -\sum_i c_i^2 \log_2 c_i^2 = -\mathrm{Tr}\left(\rho_A \log_2 \rho_A\right) = -\mathrm{Tr}\left(\rho_B \log_2 \rho_B\right)$$

where $\rho = \psi\psi^*$ and $\rho_A = \mathrm{Tr}_{H_B}\,\rho = \sum_i c_i^2 \phi_{A,i}\phi_{A,i}^*$ (Shannon and von Neumann entropy, reduced density matrices). Below ψ will be the (critical) ground state of some physical system.

Known results

$1D$ spin chain systems

EE for a block of L spins and the rest of the system

- Vidal, Latorre, Rico, Kitaev '03: XXZ and XY chains
- Jin, Korepin '04: XX chain, asymptotics for EE using a Hartwig–Fisher type theorem for Toeplitz determinants
- Its, Jin, Korepin '05: XY chain, asymptotics for EE using Riemann–Hilbert techniques

Recall

$$H_{XY} = \sum_k (1+\gamma)\sigma_k^x \sigma_{k+1}^x + (1-\gamma)\sigma_k^y \sigma_{k+1}^y + \lambda \sigma_k^z$$

$\gamma = 0$: XX model
$\gamma = 1$: Ising chain
$\lambda \in (-2, 2)$ (for $|\lambda| > 2$: noncritical ferromagnetic state) Important for us result from Jin, Korepin '05:

$$S_{XX}(L) = \frac{1}{3}\log_2 L + o(\log_2 L), \qquad L \to \infty.$$

The constant term was also computed in their work. The proof utilized a certain Fisher–Hartwig type asymptotic formula for a Toeplitz determinant corresponding to a singular symbol proved in Basor '79. (Self-correlations for the translation invariant system above are described by a Toeplitz matrix.)

Known results for $D > 1$: Entropic Area Law

It has been known that in various higher dimensional systems the entropy of a subsystem of size L grows as the area of the boundary:

$$S(L) \sim L^{D-1}, \qquad L \to \infty.$$

- Bombelli, Koul, Lee, Sorkin '86: suggested the entropy of a scalar field resticted to a subsystem to be a quantum contribution to the Bekenstein–Hawking entropy
- Srednicki '93: found the area law for the entropy of a massless scalar field (e.g., acoustic modes of a lattice)
- Plenio, Eisert, Dreissig, Cramer '05: proved the latter statement for the harmonic lattice, $H = \frac{pp^T}{2} + \frac{xVx^T}{2}$, nearest neighbor interaction

- Hamma, Ionicioiu, Zanardi '05: area law for the Kitaev model. In that model, a lattice on $\mathbb{T}^D$ is considered and

  $$H = -\sum_s a_s - \sum_p b_p$$

  where $a_s = \prod_{j \in s} \sigma_j^x$ (star), $b_p = \prod_{j \in p} \sigma_j^z$ (plaquette).

New results

We find the simplest example of a system that violates the entropic area law: a D-dimensional system of free fermions on a lattice or in the continuum.

- G., Klich '06 (Phys. Rev. Lett. 96 (100503), 2006)
- G., math.FA/0212215
- G., Klich, Laptev, in progress

Consider

$$H = \int d^D k \, \epsilon(k) \, \psi_k^\dagger \psi_k.$$

The ground state at $T = 0$ (critical system) is the filled Fermi sea $\Gamma = \{k | \epsilon(k) \leq \epsilon_F\}$. Assume Γ is compact and let Ω be a compact set in the real space.

We study EE between the fermions located in $L\Omega$ and in its complement, as $L \to \infty$. More precisely:

In the lattice (continuous) case the setup is as follows:

- $\Gamma \subset \mathbb{T}^D$ $\qquad$ ($\mathbb{R}^D$)
- $\Omega \subset \mathbb{Z}^D$ $\qquad$ ($\mathbb{R}^D$)
- $Q = \chi_{L\Omega}$ acts in $L^2(\mathbb{R}^D)$ $\qquad$ ($l^2(\mathbb{Z}^D)$)
- $P = \mathcal{F}\chi_\Gamma\mathcal{F}^{-1}$ acts in $L^2(\mathbb{R}^D)$ $\qquad$ ($l^2(\mathbb{Z}^D)$)
- $\mathcal{F}$ is the Fourier transform $\qquad$ (Fourier series)

It turns out that all the important quantities describing the system of free fermions can be written in terms of PQP:

It was shown in Klich '05 that

- average number of fermions $= \operatorname{Tr} PQP$
- variance in the fermionic number $(\Delta N)^2 = \operatorname{Tr} PQP(I - PQP)$
- $S(L) = h(PQP)$ where $h(t) = -t \log_2 t - (1-t) \log_2 (1-t)$

We are ready to formulate the results:

Two-sided estimates for EE: box-like domains

Let $\Gamma = [-\frac{1}{2}, \frac{1}{2}]^D$ in both cases.

Let $\Omega = \otimes_{j=1}^{D} \{0, \cdots, L_j\} \qquad (\otimes_{j=1}^{D} [0, L_j])$. Then

$$\frac{1}{2D} \sum_{j=1}^{D} S_1(L_j) \prod_{i \neq j} L_i \leq S(L_1, \cdots, L_D) \leq \sum_{j=1}^{D} S_1(L_j) \prod_{i \neq j} L_i$$

where $S_1(L)$ denotes the EE for a subsystem of size L of a $1D$ system. This, together with the above mentioned result of Jin, Korepin '05 that $S_1(L) \sim \frac{1}{3} \log_2 L$ for the discrete setting, implies in particular the following:

For a pair of cubic domains in the discrete case there are $C_1, C_2 > 0$ so that

$$C_1 L^{D-1} \log_2 L \leq S(L) \leq C_2 L^{D-1} \log_2 L, \qquad L \to \infty,$$

which proves the violation of the Entropic Area Law for the system under consideration. For unions of cubes: same estimates follow from $S(\cup_i C_i) \leq \sum_i S(C_i)$ proved by Lieb, Ruskai '73.

We are working on the proof that

$$S_1(L) = \frac{1}{3}\log_2 L + o(\log_2 L), \qquad L \to \infty,$$

also holds in the continuous case (Riemann–Hilbert analysis, sine kernel). This together with the mentioned estimate

$$\frac{1}{2D}\sum_{j=1}^{D} S_1(L_j)\prod_{i\neq j} L_i \le S(L_1, \cdots, L_D) \le \sum_{j=1}^{D} S_1(L_j)\prod_{i\neq j} L_i$$

would imply that the estimate

$$C_1 L^{D-1}\log_2 L \le S(L) \le C_2 L^{D-1}\log_2 L, \qquad L \to \infty,$$

also holds for cubic domains in the continuous case (and hence the entropic area law does not hold for such systems).

Two-sided estimates for EE: general domains

Recall $(\Delta N)^2 = \operatorname{Tr} PQP(I - PQP)$. For any compact Ω, Γ in both lattice and continuous case we have

$$4(\Delta N)^2 \leq S(L) \leq (\Delta N)^2 \, O(\log_2 L), \qquad L \to \infty.$$

This was proved in the lattice case for $D = 1$ in Fannes, Haegeman, Mosonyi '03.

The proof for $D > 1$ in the lattice case is the same, but in the continous case we need a new idea (and then use operator monotonicity of the square root Löwner '34 and estimates on singular numbers Birman, Solomjak '77).

We will now discuss the Widom conjecture and then continue with our results.

The Widom Conjecture

Let Γ, Ω be two compact sets in $\mathbb{R}^D$ with smooth boundaries and let f be analytic on $\{z : |z| < r\}$, $r > 1$, and $f(0) = 0$. Then for P, Q as before

$$\begin{aligned}
\operatorname{Tr} f(PQP) = {} & \left(\frac{L}{2\pi}\right)^{D} f(1) \int_\Omega \int_\Gamma dx\, dk \\
& + \left(\frac{L}{2\pi}\right)^{D-1} \frac{\ln L}{4\pi^2} U(f) \int_{\partial\Omega} \int_{\partial\Gamma} |n_x \cdot n_k|\, dS_x\, dS_k \\
& + o(L^{D-1} \ln L), \qquad L \to \infty,
\end{aligned}$$

where

$$U(f) = \int_0^1 \frac{f(t) - t f(1)}{t(1-t)}\, dt$$

and n_x, n_k are the unit normals.

Partial results towards proving the conjecture:

- Landau, Widom '80: $D = 1$, finite unions of intervals
- Widom '82: $D = 1$, additional smooth dependence on x, k
- Widom '90: $D > 1$, one set is a half-space and smooth decay
- G., '02: $D > 1$, one-term version with sharp remainder, fractal type sets allowed

Back to the results for EE:

Particle fluctiation asymptotics: verification of the conjecture for $f(t) = t(1-t)$

If Γ, Ω are two compact sets in $\mathbb{R}^D$ with smooth boundaries then for P, Q as above

$$(\Delta N)^2 = \frac{L^{D-1} \log_2 L}{(2\pi)^{D-1}} \frac{\ln 2}{4\pi^2} \int_{\partial\Omega} \int_{\partial\Gamma} |n_x \cdot n_k| \, dS_x \, dS_k \\ + o(L^{D-1} \log_2 L), \qquad L \to \infty.$$

G., Klich, Laptev '06 This together with the mentioned above estimate

$$4(\Delta N)^2 \leq S(L) \leq (\Delta N)^2 \, O(\log_2 L), \qquad L \to \infty,$$

implies:

In the continuous case (smooth boundaries, general shapes):

$$C_1 L^{D-1} \log_2 L \leq S(L) \leq C_2 L^{D-1} (\log_2 L)^2, \qquad L \to \infty,$$

for some $C_1, C_2 > 0$ that depend on Ω, Γ. The above two-sided estimate was found independently in Wolf '06 for cubic domains in the lattice case. As mentioned above for this case we have a correct order estimate

$$C_1 L^{D-1} \log_2 L \leq S(L) \leq C_2 L^{D-1} \log_2 L, \qquad L \to \infty.$$

Explicit formula for EE based on the Widom conjecture

Recall $S(L) = \operatorname{Tr} h(PQP)$ where

$$h(t) = -t \log_2 t - (1-t) \log_2 (1-t).$$

Note that h is not analytic, so the conjecture does not apply to h.
Formally applying the conjecture and noting that

$$h(1) = 0, \qquad U(h) = \frac{\pi^2}{3} \ln 2$$

we obtain

$$S(L) = \left(\frac{L}{2\pi}\right)^{D-1} \frac{\log_2 L}{12} \int_{\partial\Omega} \int_{\partial\Gamma} |n_x \cdot n_k| \, dS_x \, dS_k$$
$$+ o(L^{D-1} \log_2 L), \qquad L \to \infty.$$

▮

1. This asymptotic formula is consistent with what is expected in the continuous case for $D = 1$: $\frac{1}{12} \cdot 4 = \frac{1}{3}$ (for $1D$ lattice: $S(L) \sim \frac{1}{3} \log_2 L$) ▮
2. The above asymptotic formula was checked numerically independently in
   - Barthel, Chung, Schollwoeck, cond-mat/0602077 ($D = 2$)
   - Li, Ding, Yu, Roscilde, Haas, quant-ph/0602094 ($D = 2, 3$)

Strong Szegö Limit Theorem (SSLT)

Let $b(\theta) \geq \epsilon > 0$ be a smooth function on $\mathbb{T}^1$ and consider the operators

- P_n: projection on the span of $\{e^{ik\theta}\}_{|k|\leq n}$
- B: multiplication by b

acting in $L^2(\mathbb{T}^1)$. The operator B is represented in the basis $\{e^{ik\theta}\}_{k\in\mathbb{Z}}$ by a Toeplitz matrix formed using the Fourier coefficients $(\widehat{b}_{j-k})_{j,k\in\mathbb{Z}}$.

The following holds

$$\begin{aligned}\ln\det P_nBP_n &= (2n+1)\int_0^{2\pi}\ln b(\theta)\,\frac{d\theta}{2\pi}\\ &\quad + \sum_{k=1}^{\infty} k(\widehat{\ln b})_k(\widehat{\ln b})_{-k} + o(1), \qquad n\to\infty.\end{aligned}$$

The above asymptotic formula can be rewritten as

$$\mathrm{Tr}\,\ln(P_nBP_n) = \mathrm{Tr}\,P_n\ln(B)P_n + \text{a lower order correction}$$

as $n\to\infty$. Note that the Widom conjecture can also be written as

$$\mathrm{Tr}\,f(PQP) = \mathrm{Tr}\,Pf(Q)P + \text{a lower order correction}$$

as $L\to\infty$, and hence it is a higher dimensional (continuous) version of the SSLT.

Onsager's computation of the spontaneous magnetization for the $2D$ Ising model: a similarity with the computation of EE for fermions

Recall

$$
\begin{aligned}
Z &= \sum_{\sigma=\pm 1} \prod_{\text{n.n.}} \exp\left(K_1 \sigma_{\alpha,\beta}\sigma_{\alpha,\beta+1} + K_2 \sigma_{\alpha,\beta}\sigma_{\alpha+1,\beta}\right) \\
\kappa(T) &= \left(\sinh\frac{2}{kK_1T}\sinh\frac{2}{kK_2T}\right)^{-1} \\
M^2 &= \lim_{m\to\infty} <\sigma_{1,1}, \sigma_{1,1+m}>.
\end{aligned}
$$

It was shown using the SSLT that

$$M = \begin{cases} (1-\kappa(T))^{1/8}, & T < T_c \\ 0, & T > T_c. \end{cases}$$

The finite-m correlation $<\sigma_{1,1}, \sigma_{1,1+m}>$ can be written as a Toeplitz deteminant but the symbol b is such that $\int_0^{2\pi} \log b(\theta)\, d\theta = 0$. So the subleading asymptotic term in the SSLT was needed. The same situation occurs in the computation of EE: $h(1) = 0$ and so the subleading term in the Widom formula is needed.

Two-sided estimates for EE: fractal sets

It was shown in G., '02 that for any $\beta \in (0,1)$ there exist two compact sets $\Gamma, \Omega \subset \mathbb{R}^D$ so that for some $c_1, c_2 > 0$

$$c_1 L^{D-\beta} \leq (\Delta N)^2 \leq c_2 L^{D-\beta}, \qquad L \to \infty.$$

This together with the mentioned above estimate

$$4(\Delta N)^2 \leq S(L) \leq (\Delta N)^2 O(\log_2 L), \qquad L \to \infty,$$

implies that for any $\beta \in (0,1)$ there are two sets $\Gamma, \Omega \subset \mathbb{R}^D$ and two constants $C_1, C_2 > 0$ so that

$$C_1 L^{D-\beta} \leq S(L) \leq C_2 L^{D-\beta} \log_2 L, \qquad L \to \infty.$$

In Fannes, Haegeman, Mosonyi '03, a similar fractal set was constructed independently for $D = 1$. It was used in the context of estimating the EE.