I. Introduction

1. A New Method for Inverting Integrals

- Attenuated Radon Transform (SPECT)
- D to N map $F(k) = \int_0^T e^{k^2 t + ikl(t)} f(t)dt, \quad k \in \mathbb{C}.$

2. A New Method for Analysing BVPs

- Evolution PDEs in $[0, \infty)$ or $[0, 1]$
- Elliptic PDEs, moving BVPs, $\cdots$.

3. Integrable Nonlinear PDEs in $4 + 2$ and $3 + 1$

- DS and KP type generalizations.

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Feature Article

Generalized Fourier Transforms, Their Nonlinearization and the Imaging of the Brain

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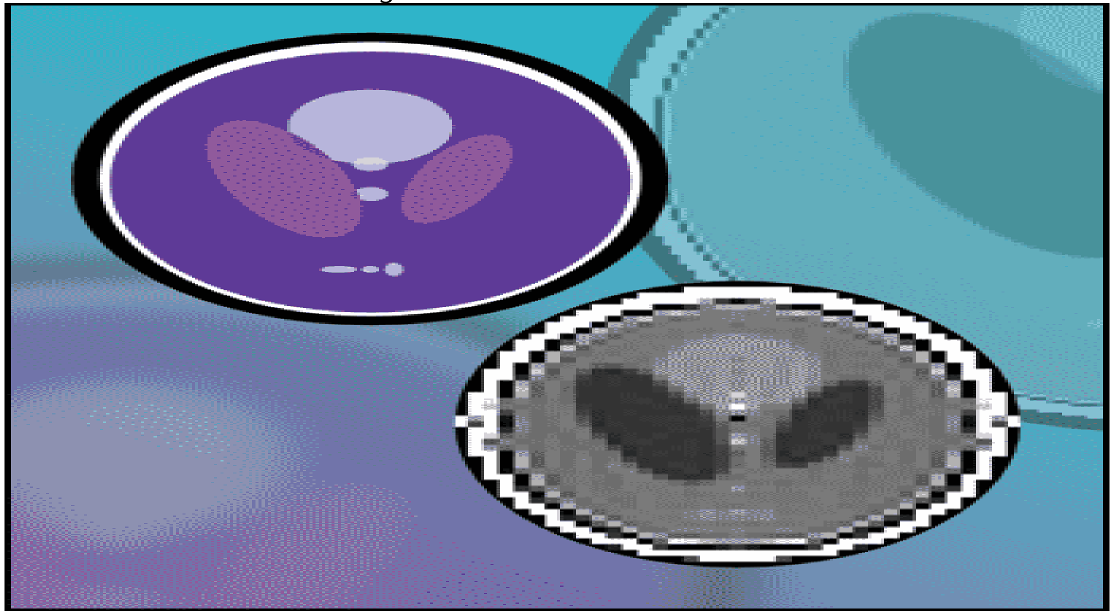


II.Inversion of Integrals-Imaging

Ablowitz-F, Beals and Coifman (1982)

$$\frac{\partial \mu}{\partial x_1} + i\sigma_3 \frac{\partial \mu}{\partial x_2} - k[\sigma_3, \mu] = Q\mu, \quad k \in \mathbb{C}, \quad (x_1, x_2) \in \mathbb{R}^2,$$

$$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad Q = \begin{pmatrix} 0 & q(x_1, x_2) \\ \bar{q}(x_1, x_2) & 0 \end{pmatrix}$$

Nonlinear FT in 2D via the $\bar{\partial}$ formalism
F-Gelfand (1992)

$$(\partial_{x_1} + i\partial_{x_2} - k)\, \mu(x_1, x_2, k) = q(x_1, x_2)$$

Novel derivation of 2D FT via $\bar{\partial}$

F-Novikov (1992)

$$\left[\frac{1}{2}\left(k+\frac{1}{k}\right)\partial_{x_1}+\frac{1}{2\mathrm{i}}\left(k-\frac{1}{k}\right)\partial_{x_2}\right]\mu(x_1,x_2,k)=f(x_1,x_2)$$

Novel derivation of Radon transform

Novikov (2003), F (2004)

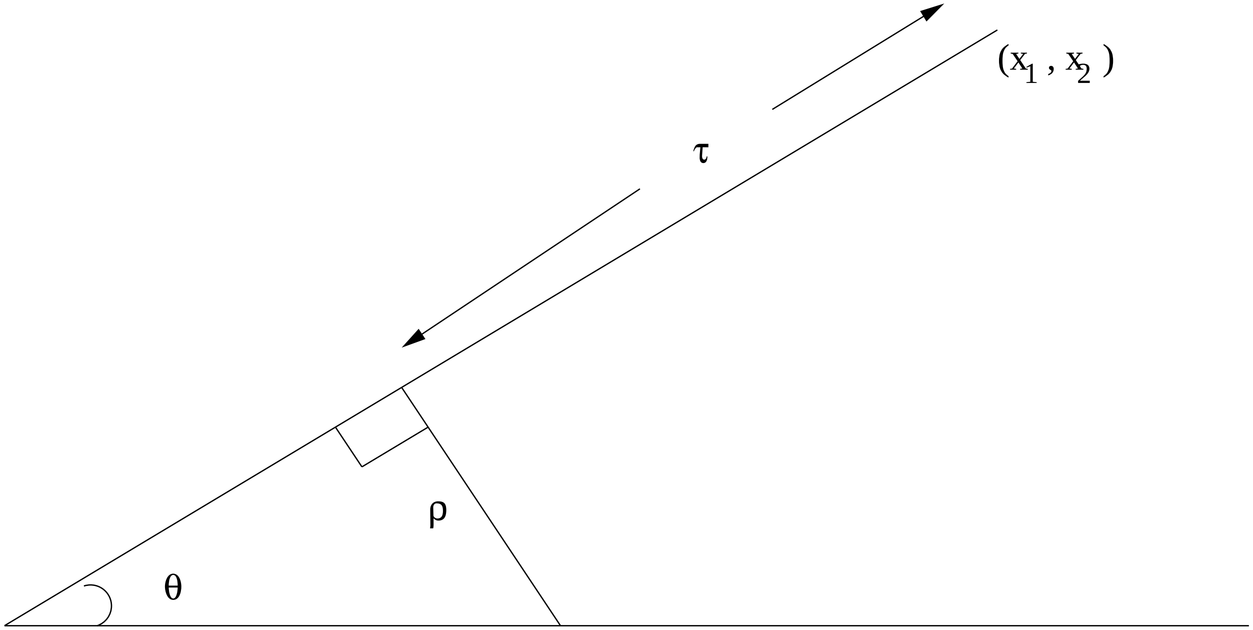
$$\left[\frac{1}{2}\left(k+\frac{1}{k}\right)\partial_{x_1}+\frac{1}{2\mathrm{i}}\left(k-\frac{1}{k}\right)\partial_{x_2}\right]\mu(x_1,x_2,k)$$

$$+f(x_1,x_2)\mu(x_1,x_2,k)=g(x_1,x_2)$$

Derivation of attenuated Radon transform

II.1 RADON TRANSFORM

Reconstruct f from its line integrals.



$$\begin{aligned} x_1 &= \tau\cos\theta - \rho\sin\theta \\ x_2 &= \tau\sin\theta + \rho\cos\theta \end{aligned} \quad \leftrightarrow \quad \begin{aligned} \tau &= x_1\cos\theta + x_2\sin\theta \\ \rho &= -x_1\sin\theta + x_2\cos\theta \end{aligned}$$

$$F(\tau, \rho, \theta) = f(\tau \cos\theta - \rho \sin\theta, \tau \sin\theta + \rho \cos\theta).$$

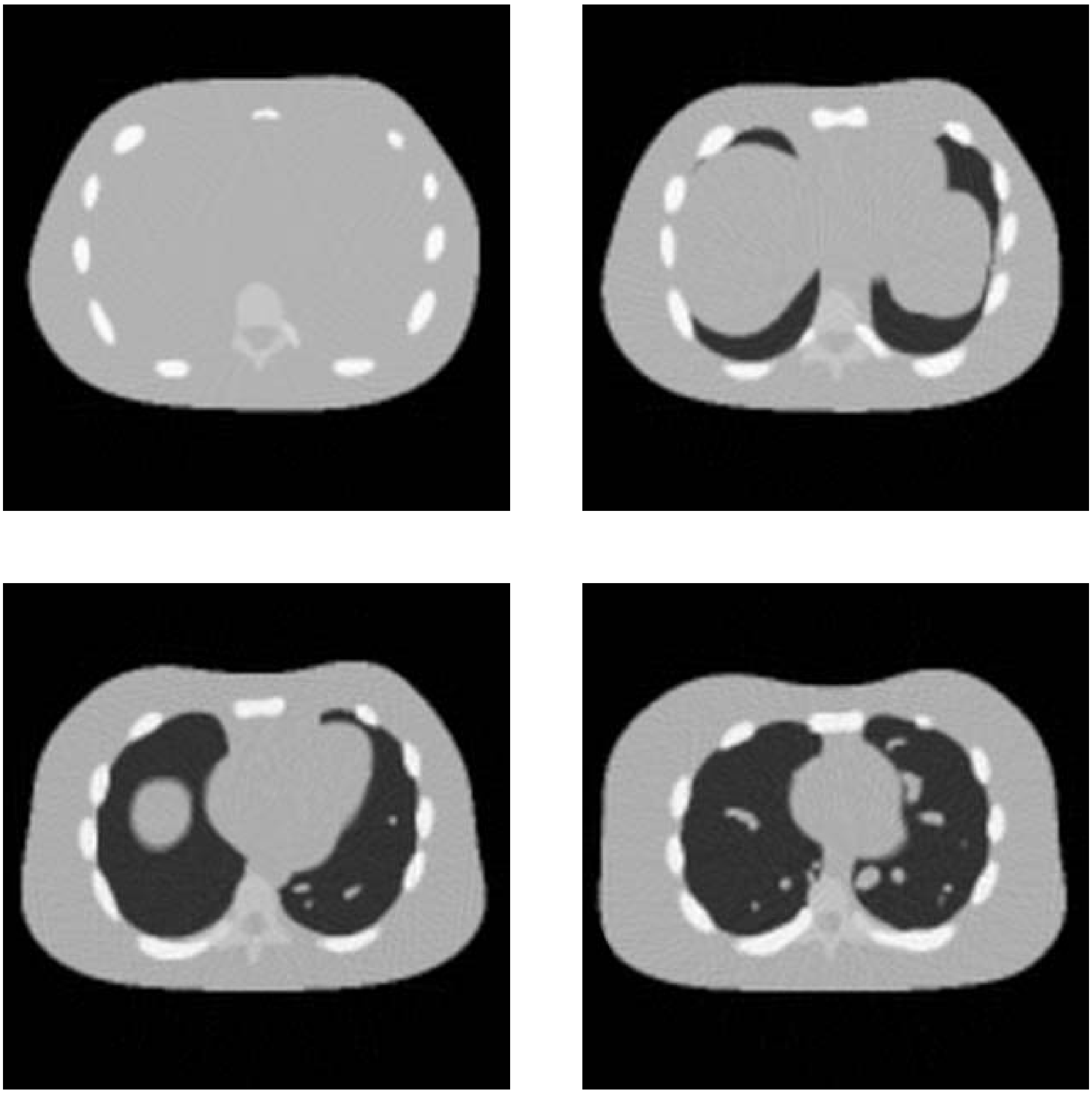
Direct Radon transform

$$\hat{f}(\rho, \theta) = \int_{-\infty}^{\infty} F(\tau, \rho, \theta) \mathrm{d}\tau.$$

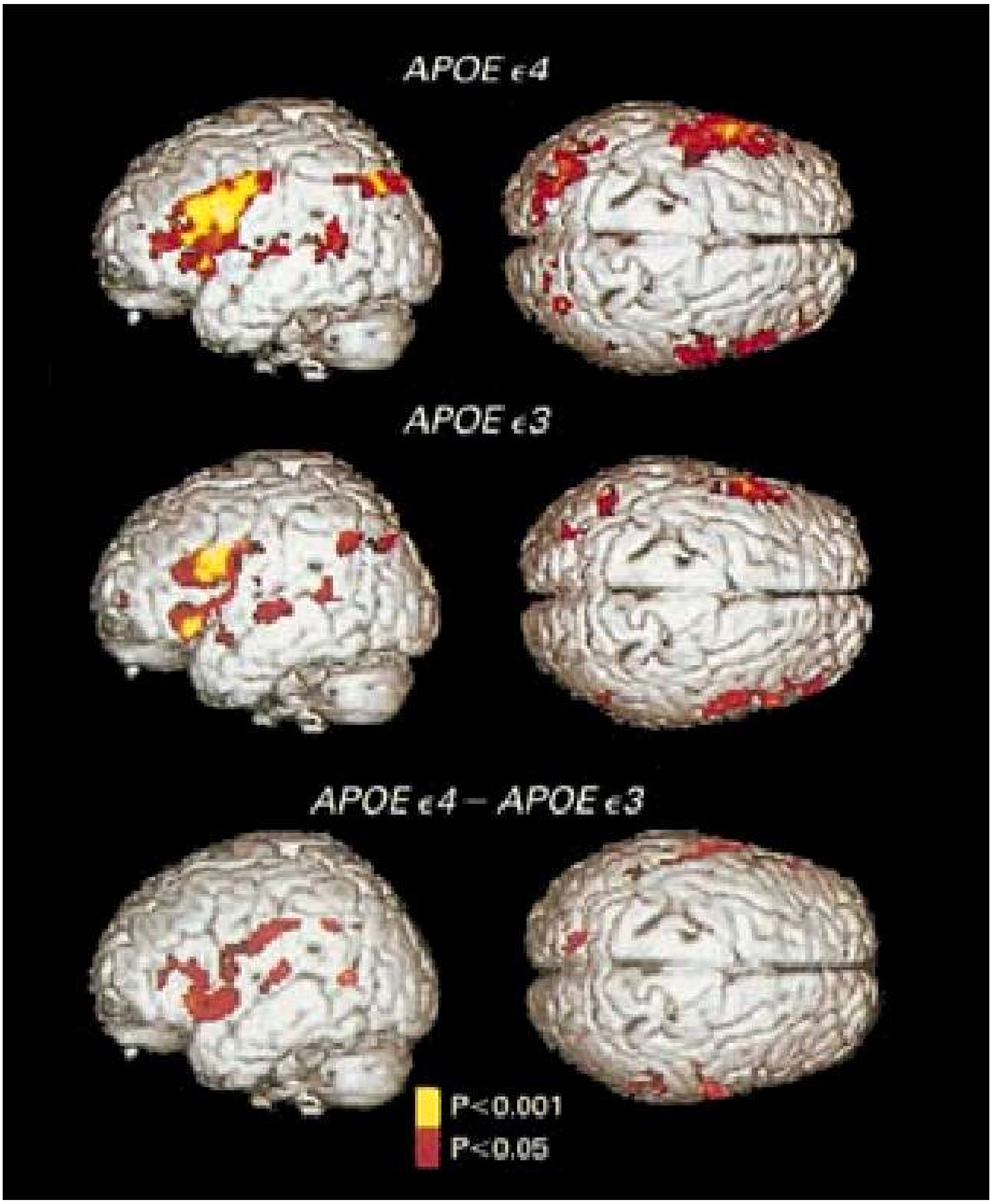
Inverse Radon transform(Filter Back Projection)

$$f(x_1, x_2) = \frac{1}{4\pi}(\partial_{x_1} - i\partial_{x_2}) \int_0^{2\pi} e^{i\theta} H(\theta, -x_1 \sin\theta + x_2 \cos\theta) \mathrm{d}\theta,$$

$$H(\theta, \rho) = \frac{1}{i\pi} \oint_{-\infty}^{\infty} \frac{\hat{f}(\rho', \theta)}{\rho' - \rho} \mathrm{d}\rho'.$$



The reconstruction of the phantoms before the filtering procedure.


APOE ε4
APOE ε3
APOE ε4 − APOE ε3
🟨 P<0.001
🟥 P<0.05

II.2 MATHEMATICS OF SPECT

$$I = \int_L e^{-\int_{L(x)} f ds} g d\tau.$$

Reconstruct g from its weighted line integrals

Direct Attenuated Radon transform

$\hat{g}_f(\rho,\theta) = \int_{-\infty}^{\infty} e^{-\int_{\tau}^{\infty} F(s,\rho,\theta)ds} G(\tau,\rho,\theta) d\tau$

Inverse Attenuated Radon transform

$g(x_1,x_2) = \frac{1}{4\pi}(\partial_{x_1} - i\partial_{x_2})\cdot$

$\cdot\int_0^{2\pi} e^{i\theta} H(\theta, x_1\cos\theta + x_2\sin\theta, x_2\cos\theta - x_1\sin\theta) d\theta$

where

$$H(\theta,\tau,\rho)=e^{\int_{\tau}^{\infty}F(s,\rho,\theta)ds}\Big\{e^{P^{-}\hat{f}(\rho,\theta)}P^{-}e^{-P^{-}\hat{f}(\rho,\theta)}+$$

$$+e^{-P^{+}\hat{f}(\rho,\theta)}P^{+}e^{P^{+}\hat{f}(\rho,\theta)}\Big\}\,\hat{g}_{f}(\rho,\theta)$$

and $P^{\pm}$ are the usual projectors

$$P^{\pm}g(\rho) = \pm\frac{g(\rho)}{2} + \frac{1}{2i\pi}\oint_{-\infty}^{\infty}\frac{g(\rho')}{\rho'-\rho}d\rho'.$$

Spectral analysis of a SINGLE equation→ Analytic inversion of integrals

$$\left\{\frac{1}{2}\left(k+\frac{1}{k}\right)\partial_{x_1}+\frac{1}{2i}\left(k-\frac{1}{k}\right)\partial_{x_2}\right\}\mu(x_1,x_2,k)=f(x_1,x_2)$$

$$k\in\mathbb{C},\quad (x_1,x_2)\in\mathbb{R}^2,\quad f(x)\in S(\mathbb{R}^2)$$

We look for μ such that $\mu = O\left(\frac{1}{z}\right), \quad z \to \infty$.

$$\boxed{\mu(x_1, x_2, k) = \frac{1}{2\pi i}\,\mathrm{sgn}\left(\frac{1}{|k|^2} - |k|^2\right)\iint_{\mathbb{R}^2} \frac{f(x_1', x_2')dx_1'dx_2'}{z' - z}, \ \ |k| \neq 1}$$

$$z \doteqdot \frac{1}{2i}\left(k - \frac{1}{k}\right)x_1 - \frac{1}{2}\left(k + \frac{1}{k}\right)x_2$$

$$\boxed{\mu(x_1, x_2, k) = -\frac{1}{2i\pi^2}\int_0^{2\pi}\frac{e^{i\theta}}{e^{i\theta} - k}\left(\oint_{-\infty}^{\infty} \frac{\hat{f}(\rho, \theta)\mathrm{d}\rho}{\rho - (x_2 \cos\theta - x_1 \sin\theta)}\right)\mathrm{d}\theta}$$

III. A new method for BVP's

III.1 Linear Problems

$$q_{xx} + q_{yy} - 4\lambda q = 0,$$

$$Q_{xx} + Q_{yy} - 4\lambda Q = 0,$$

$$(Qq_x - qQ_x)_x + \left(Qq_y - qQ_y\right)_y = 0, \quad (x, y) \in D \subset \mathbb{R}^2.$$

$$Q_{z\bar{z}} - \lambda Q = 0, \quad Q = e^{-ikz - \frac{\lambda}{ik}\bar{z}}$$

$$\left(e^{-ikz - \frac{\lambda}{ik}\bar{z}} q_z\right)_{\bar{z}} + \left(\frac{\lambda}{ik} e^{-ikz - \frac{\lambda}{ik}\bar{z}} q\right)_z = 0, \quad z \in D, \ k \in \mathbb{C}.$$

Global relation: $D \to N$ map

$$\int_{\partial D} e^{-ikz-\frac{\lambda}{ik}\bar{z}} \left(q_z dz - \frac{\lambda}{ik} q d\bar{z} \right) = 0, \quad k \in \mathbb{C}.$$

Lax pair

$$\mu_z - ik\mu = q_z, \qquad \mu_{\bar{z}} - \frac{\lambda}{ik}\mu = -\frac{\lambda}{ik} q, \quad k \in \mathbb{C}.$$

$$q_t + q_x + q_{xxx} = 0, \quad q(0,t) = g_0(t), \quad q(x,0) = q_0(x)$$

$$\omega(k) = k - k^3$$

$$\{e^{-ikx+i\omega(k)t}q\}_t - \{e^{-ikx+i\omega(k)t}[(k^2-1)q - ikq_x - q_{xx}]\}_x = 0$$

$$\mu_x - ik\mu = q$$

$$\mu_t + i\omega(k)\mu = (k^2-1)q - ikq_x - q_{xx}$$

1. Spectral Analysis ⟶ Two Maps

(A) Boundary Values ⟶ Spectral Functions

$$\{q_0(x), g_0(t), q_x(0,t), q_{xx}(0,t)\} \longrightarrow \{\hat{q}_0(k), \tilde{g}(k)\}$$

$$\hat{q}_0(k) = \int_0^\infty e^{-ikx} q_0(x)dx, \quad \mathrm{Im}k \leq 0$$

$$\widetilde{g}(k) = (k^2-1)\widetilde{g}_0(\omega(k)) - ik\widetilde{g}_1(\omega(k)) - \widetilde{g}_2(\omega(k)), \quad k \in \mathbf{C},$$

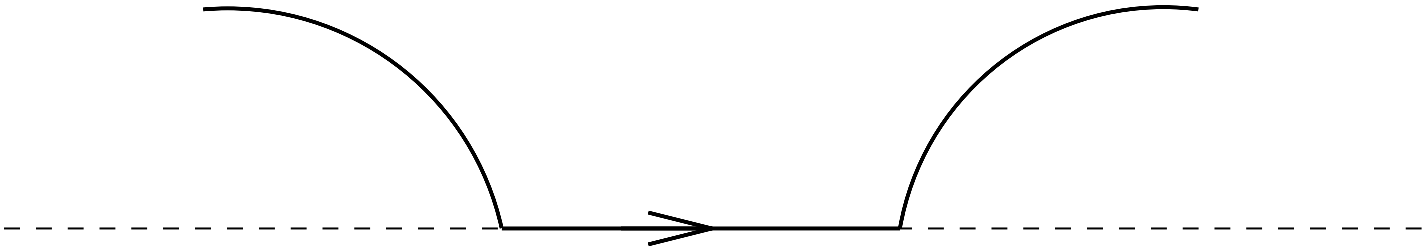
$$\widetilde{g}_0 = \int_0^T e^{i\omega(k)t} g_0(t)dt,$$

$$\widetilde{g}_1 = \int_0^T e^{i\omega(k)t} q_x(0,t)dt, \quad \widetilde{g}_2 = \int_0^T e^{i\omega(k)t} q_{xx}(0,t)dt.$$

(B) $\{\hat{q}_0(k), \tilde{g}(k)\} \longrightarrow q(x,t)$

$$q(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx - i\omega(k)t}\hat{q}_0(k)dk + \frac{1}{2\pi}\int_{\partial D} e^{ikx - i\omega(k)t}\tilde{g}(k)dk$$

$$D : \{k \in \mathbb{C}, \quad \mathrm{Im}k > 0, \quad \mathrm{Im}\omega(k) > 0\}.$$



2. Global Relation $\longrightarrow$ D to N Map

$$\tilde{g}_2 + ik\tilde{g}_1 = (k^2 - 1)\tilde{g}_0 - \hat{q}_0(k), \quad \mathrm{Im}k \leq 0$$

$$\omega(k) = \omega(\nu(k)), \quad \nu - \nu^3 = k - k^3, \quad \nu^2 + \nu k + k^2 = 1$$

$$\tilde{g}_1 = -i(\nu_1 + \nu_2)\tilde{g}_0 + i\frac{\hat{q}_0(\nu_1) - \hat{q}_0(\nu_2)}{\nu_1 - \nu_2},$$

$$\tilde{g}_2 = -(1 + \nu_1\nu_2)\tilde{g}_0 + \frac{\nu_2\hat{q}_0(\nu_1) - \nu_1\hat{q}_0(\nu_2)}{\nu_1 - \nu_2}$$

Theorem (F and Sung)
Let $q(x,t)$ satisfy

$$\begin{aligned} q_t + q_x + q_{xxx} &= 0, \quad 0 < x < \infty, \quad 0 < t < T, \\ q(x,0) &= q_0(x) \in H^1(R^+) \\ q(0,t) &= g_0(t) \in H^1(0,T), \quad g_0(0) = q_0(0). \end{aligned}$$

Then

$$q(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx-i\omega(k)t}\hat{q}_0(k)dk + \frac{1}{2\pi}\int_{\partial D} e^{ikx-i\omega(k)t}[(1-3k^2)\tilde{g}_0]$$

$$+\frac{\nu_1 - k}{\nu_2 - \nu_1}\hat{q}_0(\nu_2) + \frac{\nu_2 - k}{\nu_1 - \nu_2}\hat{q}_0(\nu_1)]dk$$

- Lax pair: Deeper kind of separability
- **No** transform in x
- Heat equation: Uniformly convergent representation
- New numerical schemes
- Beyond separation of variables
- Spectral origin and effective implementation of the "Fundamental Principle"

III.2 Nonlinear Problems

Direct Map: $q_0(x) \to \widehat{q_0}(k), \quad \{q(0,t), q_x(0,t), q_{xx}(0,t)\} \to \widetilde{g}(k)$
Inverse Map: *scalar* RH problem.

Direct Map:

$$\widehat{q_0}(k) \to \begin{pmatrix} \bar{a}(k) & b(k) \\ & \\ b(k) & a(k) \end{pmatrix}, \quad \widetilde{g}(k) \to \begin{pmatrix} \overline{A(\bar{k})} & B(k) \\ & \\ \overline{B(\bar{k})} & A(k) \end{pmatrix}$$

Inverse Map: *Matrix* RH problem involving $\exp[ikx - i\omega(k)t]$ and

$$\boxed{\gamma(k) = \frac{b(k)}{\bar{a}(k)}} \quad \boxed{\Gamma(k) = \frac{1}{a(k)\left(\frac{\bar{A}(\bar{k})}{\bar{B}(\bar{k})}a(k) - b(k)\right)}}$$

Simple problems: solve global relation by *algebra*. In general: solve the global relation by *inverting integrals*
$q_t - q_{xxx} + qq_x = 0$

$$q(0,t) = \chi, \quad q_{xx}(0,t) = \chi + 3x^2, \quad \frac{B(k)}{A(k)} = \frac{f(k)b(\nu(k)) - a(\nu(k))}{f(k)a(\nu(k)) - b(\nu(k))},$$

$$\nu^2 + k\nu + k^2 + \frac{1}{4} = 0, \quad f(k) = \frac{\nu + k}{\nu - k}\left(1 - \frac{4\nu k}{\chi}\right).$$

General case

The spectral functions are defined by

$$A(k) = e^{2ik^2T}\bar{\Phi}_2(T,\bar{k}), \quad B(k) = -e^{2ik^2T}\Phi_1(T,\bar{k}),$$

where $\Phi(t,k) = (\Phi_1, \Phi_2)$ satisfies :

$$\Phi_t + 2ik^2\sigma_3\Phi = (2kG_0(t) + G_1(t))\,\Phi, \quad t > 0, k \in \mathbb{C}, \Phi(0,k) = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

with

$$G_0(t) = \begin{pmatrix} 0 & g_0(t) \\ \bar{g}_0(t) & 0 \end{pmatrix}, G_1(t) = \begin{pmatrix} -|g_0(t)|^2 & g_1(t) \\ -\bar{g}_1(t) & |g_0(t)|^2 \end{pmatrix}.$$

Let $q_0(x) = 0$. Then the global relation becomes

$$\int_0^T e^{4ik^2 s}[i|g_0(s)|^2\Phi_1(s,k) - (2kg_0(s) + ig_1(s))\Phi_2(s,k)]ds = 0, \quad k \in \Omega^+$$

Ω^+ : First quadrant of complex $k-$plane.
This can be solved in CLOSED FORM!

For $q_0(x) \neq 0$:

$$g_1(t) = \frac{g_0(t)}{\pi} \int_{\partial\Omega^+} e^{-2ik^2t}(\Phi_2(t,k) - \Phi_2(t,-k))dk$$

$$+\frac{4i}{\pi} \int_{\partial\Omega^+} e^{-2ik^2t} k \frac{b(k)}{a(k)} \bar{\Phi}_2(t,\bar{k})dk$$

$$+\frac{2i}{\pi} \int_{\partial\Omega^+} \Big(e^{-2ik^2t} k[\Phi_1(t,k) - \Phi_1(t,-k)] + ig_0(t)\Big)dk,$$

- Large t asymptotics for decaying boundary conditions at $x=0$ (F, Its)

Using the Deift-Zhou method

$$q = \frac{1}{\sqrt{t}}\alpha(\xi)\exp\left[\frac{ix^2}{4t} + 2i\alpha^2(\xi)\ln t + i\Phi(\xi)\right] + o(1),\ t \to \infty,$$

$$\xi = -\frac{x}{4t} \text{ and } \frac{x}{t} = O(1),$$

where

$$\alpha^2(k) = \frac{1}{4\pi}\ln\left(1 - |\gamma(k) - \bar{\Gamma}(k)|^2\right),$$

$$\Phi(k) = -6\alpha^2(k)\ln 2 + \frac{\pi}{4} + \arg(\gamma(\bar{k}) - \bar{\Gamma}(\bar{k}))$$

$$+ \arg\mathbf{\Gamma}(2i\alpha^2(k)) - 4\int_{-\infty}^{k}\ln(\ell - l)\mathrm{d}(\alpha^2(\ell)),$$

$\mathbf{\Gamma}$: gamma function

- Large t Asymptotics for periodic in t boundary conditions at $x = 0$ (Boutet-de Monvel, Kotlyarov)
- Small dispersion limit (Kamvissis)

IV. Nonlinear PDEs in 4+2 and 3+1

(F, PRL, May 2006)

$$\frac{\partial\mu}{\partial\bar{x}}+\sigma_3\frac{\partial\mu}{\partial\bar{y}}-k[\sigma_3,\mu]+Q\mu=0, \qquad (*)$$

$$x=\frac{1}{2}(\xi+\eta), y=\frac{1}{2}(\xi-\eta), k=k_1+ik_2,$$

$$\xi=\xi_1+i\xi_2, \quad \eta=\eta_1+i\eta_2$$

$$\sigma_3=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}, \quad Q=\begin{pmatrix}0 & q_1(\xi_1,\xi_2,\eta_1,\eta_2)\\ q_2(\xi_1,\xi_2,\eta_1,\eta_2) & 0\end{pmatrix}$$

Nonlinear FT

$\{q_1, q_2\} \to \{f_1, f_2\}$

$$f_1(k_1, k_2, \lambda_1, \lambda_2) = c \int_{\mathbb{R}^4} e^{-4i(k_2\xi_1 - k_1\xi_2 + \lambda_2\eta_1 - \lambda_1\eta_2)} q_1 \mu_{22} d\xi_1 d\xi_2 d\eta_1 d\eta_2,$$

$$f_2(k_1, k_2, \lambda_1, \lambda_2) = c \int_{\mathbb{R}^4} e^{-4i(-\lambda_2\xi_1 + \lambda_1\xi_2 - k_2\eta_1 + k_1\eta_2)} q_2 \mu_{11} d\xi_1 d\xi_2 d\eta_1 d\eta_2,$$

$c = (2/\pi)^3$.
$\mu(\xi_1, \xi_2, \eta_1, \eta_2, k_1, k_2)$ is determined in terms of $\{q_j(\xi_1, \xi_2, \eta_1, \eta_2)\}_1^2$ by $(*)$ with

$$\mu \sim I \ \text{ as } \ |\xi_1|^2 + |\xi_2|^2 + |\eta_1|^2 + |\eta_2|^2 \to \infty$$

$$\{f_1, f_2\} \to \{q_1, q_2\}$$

$$q_1(\xi_1, \xi_2, \eta_1, \eta_2) = \int_{\mathbb{R}^4} e^{4i(k_2\xi_1 - k_1\xi_2 + \lambda_2\eta_1 - \lambda_1\eta_2)} f_1 \mu_{11} dk_1 dk_2 d\lambda_1 d\lambda_2$$

$$q_2(\xi_1, \xi_2, \eta_1, \eta_2) = \int_{\mathbb{R}^4} e^{4i(-\lambda_2\xi_1 + \lambda_1\xi_2 - k_2\eta_1 + k_1\eta_2)} f_2 \mu_{22} dk_1 dk_2 d\lambda_1 d\lambda_2,$$

μ is determined in terms of $\{f_j(k_1, k_2, \lambda_1, \lambda_2)\}_1^2$ by

$$\frac{\partial \mu(k_1, k_2)}{\partial \bar{k}} = \int_{\mathbb{R}^2} \mu(\lambda_1, \lambda_2) \cdot$$

$$\cdot \begin{pmatrix} 0 & e^{-4i(k_2\xi_1 - k_1\xi_2 + \lambda_2\eta_1 - \lambda_1\eta_2)} f_1 \\ e^{-4i(-\lambda_2\xi_1 + \lambda_1\xi_2 - k_2\eta_1 + k_1\eta_2)} f_2 & 0 \end{pmatrix} d\lambda_1 d\lambda_2,$$

$$\mu \sim I + O(\frac{1}{k}), \quad k \to \infty$$

Integrable PDEs in 4+2

$$\left\{q_j^{(0)}(\xi_1,\xi_2,\eta_1,\eta_2)\right\}_1^2 \longrightarrow \left\{f_j^{(0)}(k_1,k_2,\lambda_1,\lambda_2)\right\}_1^2$$

$$\left\{f_1^{(0)}E, f_2^{(0)}E^{-1}\right\} \longrightarrow \left\{q_j(\xi_1,\xi_2,\eta_1,\eta_2,t_1,t_2)\right\}_1^2$$

$$E = e^{4i(\lambda_1\lambda_2+k_1k_2)t_2-2i(\lambda_1^2-\lambda_2^2+k_1^2-k_2^2)t_1}$$

Then,

$$(-1)^j\partial_{\bar{t}}q_j + \frac{1}{4}(\partial_{\bar{\xi}}^2 + \partial_{\bar{\eta}}^2)q_{\bar{j}} - q_j\partial_{\bar{\xi}}^{-1}(q_1q_2)_{\bar{\eta}} = 0,$$

$$\partial_{\bar{\xi}}^{-1}q = -\frac{1}{\pi}\int_{\mathbb{R}^2}\frac{q(\xi_1',\xi_2')}{\xi-\xi'}d\xi_1 d\xi_2.$$

Reductions to 3+1

independence of $t_1 \leftrightarrow \lambda_1\lambda_2 + k_1k_2 = 0$

Linear limit:

$$\frac{\partial^2 q}{\partial\xi_1\partial\xi_2} + \frac{\partial^2 q}{\partial\eta_1\partial\eta_2} = 0.$$

KP type

$$\frac{\partial q}{\partial\bar{t}} = \frac{1}{4}\frac{\partial^3 q}{\partial\bar{x}^3} - \frac{3}{2}q\frac{\partial q}{\partial\bar{x}} + \frac{3}{4}\partial_{\bar{x}}^{-1}\frac{\partial^2 q}{\partial\bar{y}^2}$$

independence of $t_2 \leftrightarrow k_1^3 - \lambda_1^3 + 3\lambda_1\lambda_2^2 - 3k_1k_2^2 = 0$