Integrable Systems, Random Matrices and Applications
May 24, 2006

Polynuclear growth on a flat substrate and edge scaling of GOE eigenvalues

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- Point-to-point directed polymers and GUE random matrices
- Extension to joint distributions: the Airy process

- Point-to-line directed polymers and GOE random matrices
- Extension to joint distributions: an open question

- Recent development: the Sasamoto result
- Universality

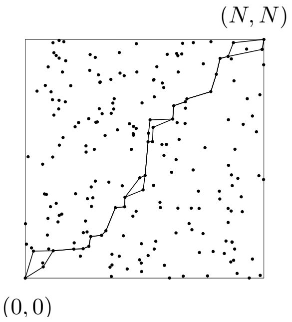
- Poisson process on $\mathbb{R}^2$ with intensity 1.
- A directed polymer π from $(0,0)$ to (N,N) is a collection of Poisson points (x_i, y_i) with $x_i < x_{i+1}$ and $y_i < y_{i+1}$.

- Length of π, $\ell(\pi)$ is the number of Poisson points in π.

- Point-to-point maximization

$\Rightarrow$ Find the maximal length

$$L_{N,N} = \max_{\pi:(0,0)\to(N,N)} \ell(\pi)$$



- Ulam's problem (poissoned version)
  Analyze the asymptotics of $L_{N,N}$ as $N \to \infty$.

- $\mathbb{E}(L_{N,N})/N \to 2$ was obtained in the 70's.

- Fluctuations live on the $N^{1/3}$ scale and are F_{GUE}-distributed [Baik, Deift, and Johansson '99]

$$\lim_{N\to\infty} \mathbb{P}(L_{N,N} \leq 2N + sN^{1/3}) = F_{\mathrm{GUE}}(s)$$

  with $F_{\mathrm{GUE}}(s)$ the Tracy-Widom distribution for the GUE ensemble of random matrices (distribution of the largest eigenvalue of $N \times N$ GUE random matrices as $N \to \infty$).

Extension to joint distributions

- In the 1d polynuclear growth model (PNG) (a growth model):

length of directed polymer $\equiv$ height function

$\Rightarrow$ Question: what about the joint distribution of

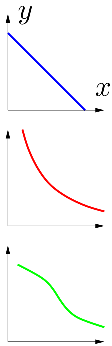
$$(L_{N,N}, L_{N+m,N-m})?$$

- PNG droplet [Prähofer, Spohn '02]: $m \sim N^{2/3}$

$$u \mapsto N^{-1/3}(L_{N+uN^{2/3},N-uN^{2/3}} - 2N) + u^2$$

converges to the Airy process $\mathcal{A}(u)$ as $N \to \infty$

- Terrace-Ledge-Kink model: along $y = N/x$ [Ferrari, Prähofer, Spohn '03]
- Generalization along any "negative-slope" path [Borodin, Olshanski '04]



- A two-matrix model
  M_0, M_t two $N \times N$ hermitian random matrices with joint distribution

$$\frac{1}{Z_{N,t}} \exp\left(-\frac{\mathrm{Tr}(M_0^2)}{2N}\right) \exp\left(-\frac{\mathrm{Tr}(M_t - qM_0)^2}{2N(1-q^2)}\right) \mathrm{d}M_0 \mathrm{d}M_t$$

  with $q = \exp(-t/2N)$
- Non-trivial correlations for $t \sim N^{2/3}$
- As $N \to \infty$, the joint distribution of the largest eigenvalues is the one of the Airy process.

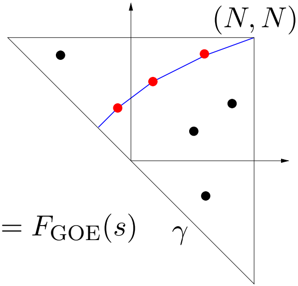
  Let $\lambda_{\max}(t)$ be the largest eigenvalue of M_t, then

$$\left(N^{-1/3}(\lambda_{\max}(0)-2N), N^{-1/3}(\lambda_{\max}(2uN^{2/3})-2N)\right) \to \left(\mathcal{A}(0), \mathcal{A}(u)\right)$$

- Another quantity of interest is the length of point-to-line directed polymers ($\equiv$ PNG growth on a flat substrate)

- Let $\gamma = \{(u,-u), u \in \mathbb{R}\}$, then the point-to-line maximal length is

$$L^{\gamma}_{N,N} = \max_{\pi:\gamma\to(N,N)} \ell(\pi)$$



- Fluctuations on a scale $N^{1/3}$, F_{GOE}-distributed

$$\lim_{N\to\infty} \mathbb{P}(L^{\gamma}_{N,N} \leq 2N + sN^{1/3}2^{-2/3}) = F_{\mathrm{GOE}}(s)$$

with F_{GOE} the GOE Tracy-Widom distribution [Baik, Rains '00]

- Open questions

Q1) Point-to-line. Let $t = uN^{2/3}$. What is the $N \to \infty$ limit of

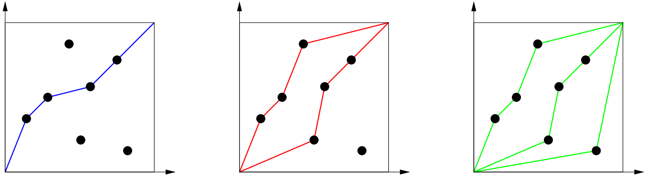
$$(N^{-1/3}(L^{\gamma}_{N,N} - 2N), N^{-1/3}(L^{\gamma}_{N+t,N-t} - 2N)) \to ???$$

Q2) Consider GOE two-matrix model. What is the limit of

$$(N^{-1/3}(\lambda_{\max}(0) - 2N), N^{-1/3}(\lambda_{\max}(2uN^{2/3}) - 2N)) \to ???$$

- The link GUE and point-to-point directed polymers extends is beyond the largest eigenvalue / directed polymers

- Let L_k be the sum of the length of the k longest distinct directed polymers, set
$\mu_1 = L_1, \quad \mu_2 = L_2 - L_1, \quad \mu_3 = L_3 - L_2, \quad \dots$



$\mu_1 = 4$ $\mu_2 = 2$ $\mu_3 = 1$

- Note: The relative positions of the points give a permutation, whose Young tableaux have rows of length μ_i [Greene '74].

- Point-to-point directed polymers: $\mu_1 \geq \mu_2 \geq \ldots \geq 0$.
- GUE eigenvalues: $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_N$

$\Rightarrow$ For any fixed k [Okounkov '00]

$$\lim_{N\to\infty} \{N^{-1/3}(\lambda_i - 2N), i = 1, \ldots, k\}$$

$$\stackrel{\mathcal{D}}{=} \lim_{N\to\infty} \{N^{-1/3}(\mu_i - 2N), i = 1, \ldots, k\}$$

- This connection extends to GUE multi-matrix and joint point-to-point directed polymers like it was for the largest eigenvalue / longest directed polymer

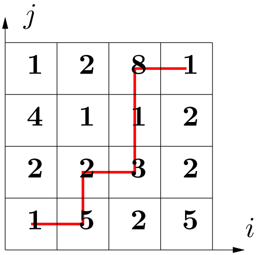
- Extension to GOE and point-to-line directed polymers [Ferrari '04]
- Conjecture: GOE multi-matrix and point-to-line directed polymers have the same limit distribution (as it is for the case GUE an point-to-point)
- Problem: In both GOE multi-matrix and point-to-line directed polymers the joint limit distribution are not known!

- Sasamoto ['05] obtained a result for a system (TASEP) which can be mapped to a discrete version of directed polymers

- Let $\omega_{i,j}$, $i, j \in \mathbb{Z}$, be i.i.d. random variables, exponentially distributed with mean 1

- A directed polymer π on $\mathbb{Z}^2$ from A to B is an up-right path connecting A to B, i.e., made up of pieces $(1,0)$ or $(0,1)$.
- Length of π: $\ell(\pi) = \sum_{(i,j)\in\pi} \omega_{i,j}$
- Maximization problem: find

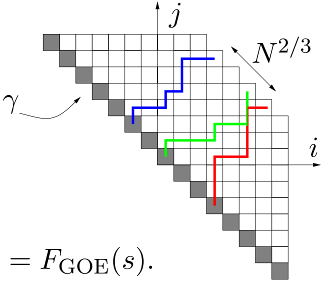
$$L_{N,N} = \max_{\pi:(0,0)\to(N,N)} \ell(\pi)$$



The $\omega_{i,j}$, a path $\pi : (0,0) \to (3,3)$
$\ell(\pi) = 21$

- Point-to-line problem: find the maximizer from the "line" $\gamma = \{(u,-u), u \in \mathbb{Z}\}$ to a point (N,N):

$$L^{\gamma}_{N,N} = \max_{\pi:\gamma\to(N,N)} \ell(\pi)$$



- Theorem (one-point):

$$\lim_{N\to\infty} \mathbb{P}(L^{\gamma}_{N,N} \leq 4N+2s(4N)^{1/3}) = F_{\mathrm{GOE}}(s).$$

For point-to-point problem [Johansson '03]: GUE Tracy-Widom distributed fluctuations

- Theorem (multi-point):
  [Borodin, Ferrari, Prähofer, Sasamoto: in preparation]
  Let $u_1 \leq u_2 \leq \ldots \leq u_k$. Then as $N \to \infty$, the joint distribution of the k directed polymers is given by the Fredholm determinant

$$\lim_{N\to\infty} \mathbb{P}\Big(\bigcap_{i=1}^{k}\{L^{\gamma}_{N-u_i(4N)^{2/3},N+u_i(4N)^{2/3}} \leq 4N + 2s_i(4N)^{1/3}\}\Big)$$
$$= \det(\mathbb{1} - \chi_s K_\infty \chi_s)_{L^2(\mathbb{R}\times\{1,\ldots,k\})}$$

  with $\chi_s(x,j) = \mathbb{1}_{[x\geq s_j]}$

- Kernel K_∞: let $B_0(x,y) = \mathrm{Ai}(x+y)$, Δ the Laplacian,

$$\begin{aligned} K_\infty(s_1,u_1;s_2,u_2) &= -(e^{(u_2-u_1)\Delta})(s_1,s_2)\mathbb{1}_{[u_2>u_1]} \\ &+ (e^{-u_1\Delta}B_0 e^{u_2\Delta})(s_1,s_2) \end{aligned}$$

- The Sasamoto kernel

$$K_\infty(s_1,u_1;s_2,u_2) = \Bigg[-\frac{1}{\sqrt{4\pi(u_2-u_1)}} \exp\left(-\frac{(s_2-s_1)^2}{4(u_2-u_1)}\right) \mathbb{1}_{[u_2>u_1]}$$
$$+ \mathrm{Ai}(s_1+s_2+(u_2-u_1)^2) \exp\left((u_2-u_1)(s_1+s_2)+\frac{2}{3}(u_2-u_1)^3\right)\Bigg]$$

⇒ a determinantal formula for the GOE Tracy-Widom distribution [Ferrari, Spohn '05]

$$F_{\mathrm{GOE}}(s) = \det(\mathbb{1} - B_s)_{L^2(\mathbb{R}_+)},$$

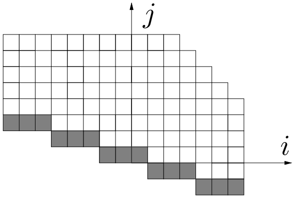
with $B_s(x,y) = \mathrm{Ai}(x+y+s)$.

- For comparison, $F_{\mathrm{GUE}}(s) = \det(\mathbb{1} - B_s^2)_{L^2(\mathbb{R}_+)}$.

A few remarks

- B_s is trace-class on $\mathbb{R}_+$.
- The Sasamoto kernel does not define a determinantal point process (positivity missing)
- One starts with Schütz formula for joint distribution of particles positions given by a Toeplitz-like determinant
- Using some special linear combination can be decomposed so that one has weight like a product of determinants (algebraic manipulations)
- This can be reinterpreted as kind of vicious walkers BUT the probabilistic interpretation is lost

- One would also expect "universality" by changing slope of the line for the point-to-line directed polymers.
- We prove it to be true for slopes $-1/d$, $d \geq 1$ [Borodin, Ferrari, Prähofer: in preparation]



The setting for $d = 3$

- Universality $\Rightarrow$ there is a common limit joint distribution for discrete and continuous directed polymers
- Proven for point-to-point case
- Conjecture that GOE multi-matrix evolves as point-to-line directed polymers
- Proven for GUE multi-matrix and point-to-point directed polymers

$\Rightarrow$ Conjecture on the evolution of the largest eigenvalue for GOE multi-matrix (analogue of the Airy process)

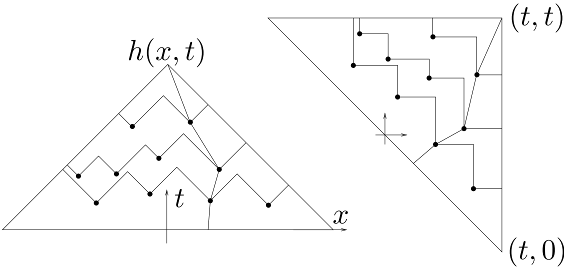
- Conjecture on the evolution of the largest eigenvalue for GOE multi-matrix:

$$\lim_{N\to\infty} \mathbb{P}\Big(\bigcap_{i=1}^{k} \{\lambda_{\max}(c_1 u_i N^{2/3}) \leq 2N + c_2 s_i N^{1/3}\}\Big)$$
$$= \det(\mathbb{1} - \chi_s K_\infty \chi_s)_{L^2(\mathbb{R}\times\{1,\ldots,k\})}$$

for some $c_1 > 0$, $c_2 = 2^{-2/3}$, and with $\chi_s(x,j) = \mathbb{1}_{[x\geq s_j]}$, $u_1 \leq u_2 \leq \ldots \leq u_k$.

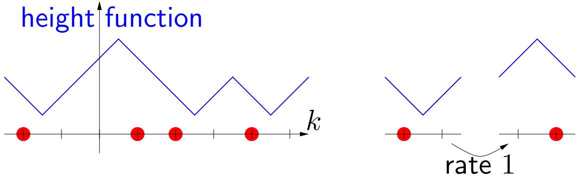
- GUE case: K_∞ is replaced by the extended Airy kernel ($c_1 = 2$ and $c_2 = 1$)

- Polynuclear growth (PNG) model. Configuration: height function $x \mapsto h(x,t) \in \mathbb{Z}$.
- The Poisson points are the space-time nucleations, a creation of a pair of up- and down-step on top of the height profile.
- Dynamics: up- (down-) steps move to the left (right) with unit speed.
  When they meet, they coalesce.



Animation

- Configurations and dynamics



- Point-to-line problem. Initial condition: only sites $-2n, n \in \mathbb{Z}$ occupied.
- $L_{M,N}$ = time needed for particle starting at $-2N$ to reach position $M - N$.