Concentration results for the β-Hermite and β-Laguerre ensembles

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joint work with **Ofer Zeitouni**

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Main thread of research: matrix models for the β-ensembles.

Main obsession: matrix models for the β-ensembles.

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Hermite model (Alan Edelman and I.D., 2001)

$$H_\beta \sim \frac{1}{\sqrt{2}} \begin{pmatrix} N(0,2) & \chi_{(n-1)\beta} & & & \\ \chi_{(n-1)\beta} & N(0,2) & \chi_{(n-2)\beta} & & \\ & \ddots & \ddots & \ddots & \\ & & \chi_{2\beta} & N(0,2) & \chi_\beta \\ & & & \chi_\beta & N(0,2) \end{pmatrix}$$

has joint eigenvalue p.d.f.

$$\propto \prod_{1 \le i < j \le n} |\lambda_i - \lambda_j|^\beta \quad e^{-\sum_{i=1}^{n} \lambda_i^2/2}$$

$$\equiv \quad |\Delta(\Lambda)|^\beta \quad e^{-\sum_{i=1}^{n} \lambda_i^2/2} .$$

Note: $\beta = 1, 2, 4$ *correspond to the Gaussian ensembles* **GOE**, **GUE**, **GSE**.

Main thread of research: matrix models for the β-ensembles.

Laguerre model (Alan Edelman and I.D., 2001)

$$L_\beta = B_\beta B_\beta^T \ ,$$

where

$$B_\beta \sim \begin{pmatrix} \chi_{2a} & & & \\ \chi_{\beta(m-1)} & \chi_{2a-\beta} & & \\ & \ddots & \ddots & \\ & & \chi_\beta & \chi_{2a-\beta(m-1)} \end{pmatrix} \ ,$$

has joint eigenvalue p.d.f.

$$\begin{aligned} &\propto \prod_{1\le i<j\le n} |\lambda_i - \lambda_j|^\beta \prod_{i=1}^m \lambda_i^{a-\beta(m-1)/2-1} \quad e^{-\sum_{i=1}^n \lambda_i/2} \\ &\equiv |\Delta(\Lambda)|^\beta \prod_{i=1}^m \lambda_i^{a-\beta(m-1)/2-1} \quad e^{-\sum_{i=1}^n \lambda_i/2} \ . \end{aligned}$$

Note: *$\beta = 1, 2, 4$ and $a = n\beta/2$ correspond to the Wishart ensembles* **real**, **complex**, **quaternion**.

Main thread of research: matrix models for the β-ensembles.

Jacobi model (Irina Nenciu and Rowan Killip, 2004; Brian Sutton, 2005)

$$J_\beta = C_\beta C_\beta^T \ ,$$

where

$$C_\beta \sim \begin{pmatrix} c_n & 0 & \dots & 0 & 0 & 0 & 0 \\ -s_n c'_{n-1} & c_{n-1} & & 0 & 0 & 0 & 0 \\ 0 & -s_{n-1}c'_{n-2} & & 0 & 0 & 0 & 0 \\ & & & \vdots & & & \\ 0 & 0 & 0 & & -s_3 c'_2 & c_2 & 0 \\ 0 & 0 & 0 & \dots & 0 & -s_2 c'_1 & c_1 \end{pmatrix} ,$$

such that

$$c_k \sim \sqrt{Beta(\frac{\beta}{2}(a+k), \frac{\beta}{2}(b+k))} \ , s_k = \sqrt{1-c_k^2} \ ,$$

$$c'_k \sim \sqrt{Beta(\frac{\beta}{2}k, \frac{\beta}{2}(k+a+b+1))} \ , s'_k = \sqrt{1-c'^2_k} \ .$$

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$$J_\beta = C_\beta C_\beta^T$$

has joint eigenvalue p.d.f.

$$\begin{aligned} &\propto \quad \prod_{1 \le i < j \le n} |\lambda_i - \lambda_j|^\beta \prod_{i=1}^{n} \lambda_i^{a_1 - \frac{\beta}{2}(n-1)-1} (1-\lambda_i)^{a_2 - \frac{\beta}{2}(n-1)-1} \\ &\equiv \quad |\Delta(\Lambda)|^\beta \prod_{i=1}^{n} \lambda_i^{a_1 - \frac{\beta}{2}(n-1)-1} (1-\lambda_i)^{a_2 - \frac{\beta}{2}(n-1)-1} . \end{aligned}$$

Note: *$\beta = 1, 2, 4$ and $a = n_1\beta/2$, $b = n_2\beta/2$ correspond to the MANOVA ensembles* **real**, **complex**, **quaternion**.

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Note: $\beta = 1, 2, 4$ *correspond to the Gaussian ensembles* **GOE**, **GUE**, **GSE**.

Main thread of research: matrix models for the β-ensembles.

Applications:

- *Practical.*

$\beta \notin \{1, 2, 4\}$ used to describe traffic flow in Germany

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Headways in traffic flow: Remarks from a physical perspective

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Traffic flow can be understood as a realization of a broad class of one dimensional physical systems, where a hard core repulsive interaction competes with a longer ranged attraction between the particles. It can be shown rigorously that the statistical properties of such systems in thermal equilibrium are well described by a family of distributions that stems from the random matrix theory. Analyzing the traffic data from different sources, we show that traffic on real roads belongs to that class of random matrix distributions. Also, various traffic simulation models show a similar behavior. It is demonstrated in such a way that the headway distribution of a highway traffic, that serves usually as a paradigm of systems driven far from equilibrium, is reasonably well described by a distribution originating from equilibrium statistical physics.

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I. RANDOM MATRIX THEORY AND TRAFFIC

Random matrix theory (RMT) appears to be a very universal instrument. Originally invented to model the energy levels of atomic nuclei it turns out to be useful in a wide range of different systems and occasions. Of special importance is the deep connection between classical chaotic systems and their quantum mechanical counterparts.

This paper reports on a work done on another connection, that between RMT and one dimensional many particle systems, with a special focus on traffic. For illustration, this relation will be discussed first with the help of the Dyson gas where it is known to be exact. The Dyson gas describes the equilibrium properties of a one dimensional system of N par-

It is well known that the statistical properties of the Dyson gas in thermal equilibrium are exactly described by RMT. In particular, for the heat bath inverse temperature $\beta=1$ or $\beta=2$ the Dyson gas conforms to the orthogonal/unitary ensemble of random matrices, respectively. In this respect two prominent statistical distributions are commonly discussed: the spacing distribution and the number variance. The spacing distribution $P(s)$ describes the probability density that two neighboring particles are found with mutual distance equal to s. The distribution $P(s)$ is scaled so that the mean distance equals one: $\langle s\rangle=\int s\,P(s)\,ds=1$. It takes into account only the two particle correlations and is therefore very robust and not sensitive to the detailed properties of the system. A more sensitive measure for discussing the correlations

$\beta \notin \{1, 2, 4\}$ used to describe traffic flow in Germany

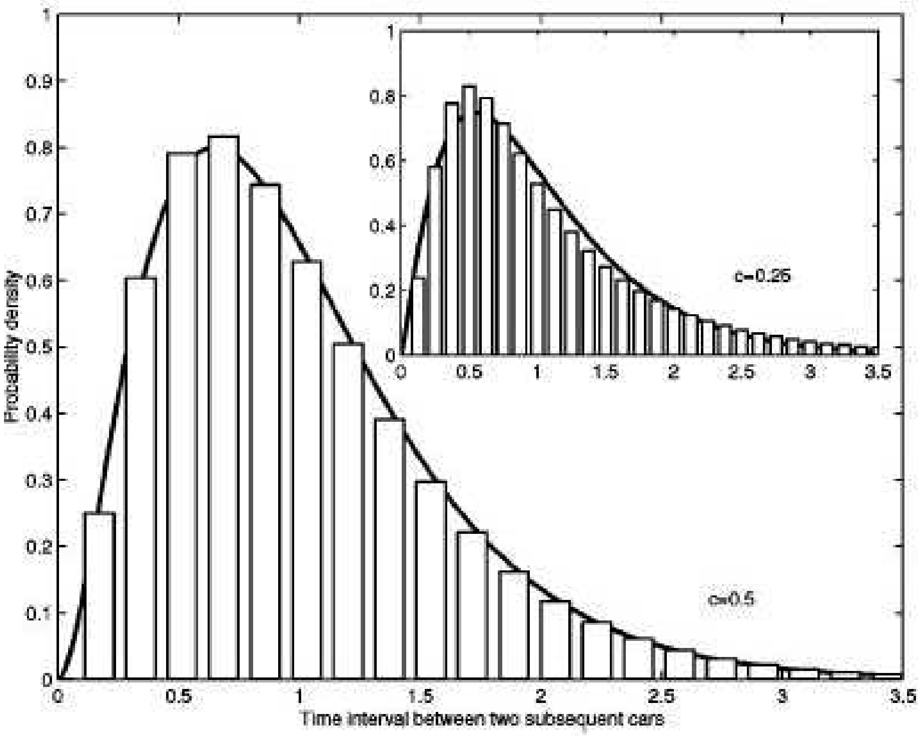


FIG. 3. Comparison between Eq. (11) and the best fit of Eq. (4). The full line represents Eq. (4) for $\beta = 1.86$ and bars display the time headway of CA model for $c = 0.5$. The same for $\beta = 1.1989$ and $c = 0.25$ is visible in the inset.

This safety conditions can be transformed into a set of update rules as follows:

$$v_{\text{safe}} = \tilde{v}(t) + 2\,b\,\frac{g(t) - \tilde{v}(t)}{2\,b + v(t) + \tilde{v}(t)}, \tag{7}$$

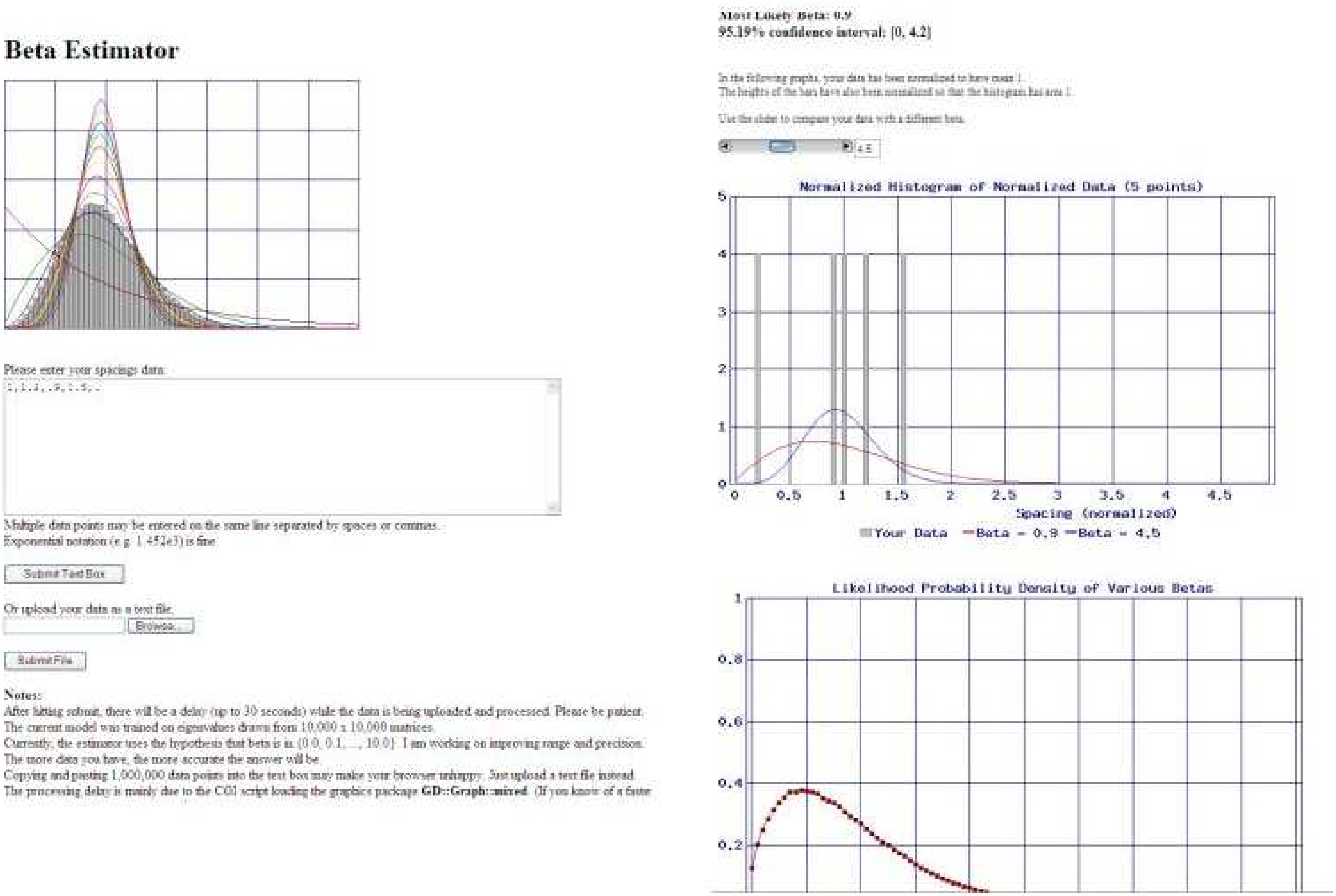
$$v_{\text{des}} = \min\{v(t) + a, v_{\text{safe}}, v_{\max}\}, \tag{8}$$

does not allow time headways smaller than one time step. Therefore it is necessary to shift the probability Eq. (11) by one timestep to the left in order to compare with the RMT result, Eq. (4). The results for $c = 0.5$, $c = 0.25$, respectively, are shown in Fig. 3. Note that the interaction between the cars of the CA model leads to a short ranged repulsion, but additionally to a medium ranged attraction (in the vicinity of $s = 1$) that probably causes the discrepancy in the time headway statistics close to the maximum of the curve in Fig. 3.

With the matrix models, we can finally sample and construct estimators!

What's my β?... courtesy of Cy Chan

http://people.csail.mit.edu/cychan/BetaEstimator.html

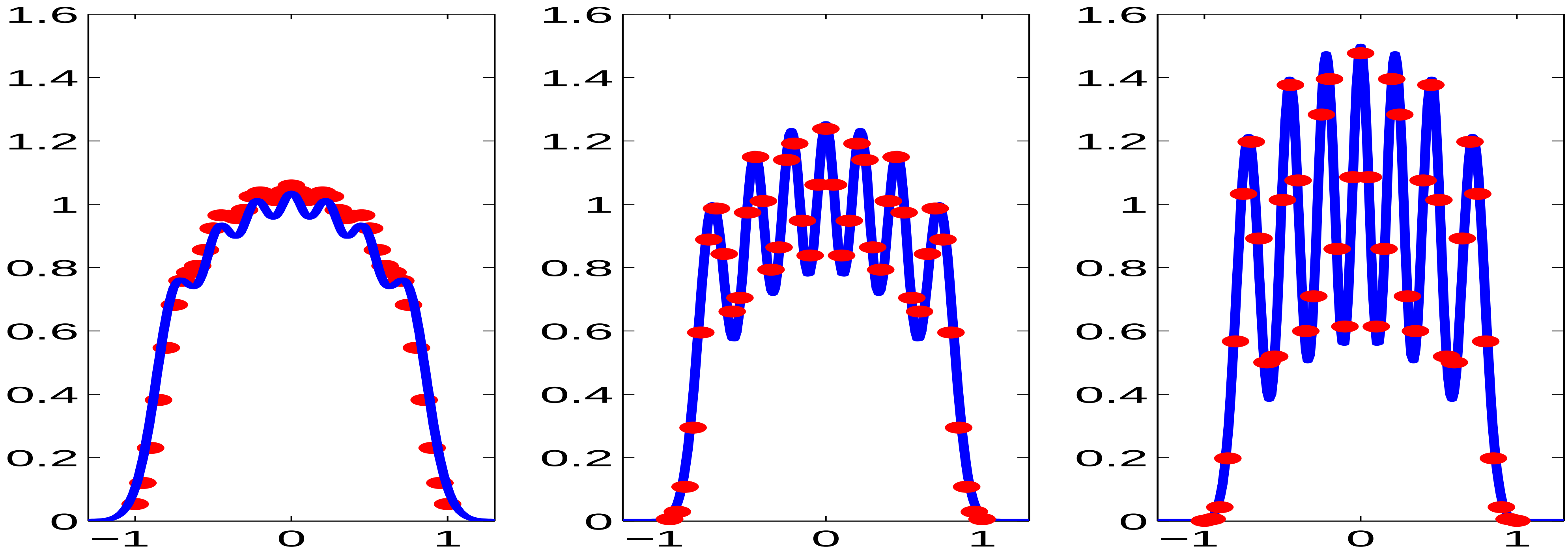


Main thread of research: matrix models for the β-ensembles.

Applications:

- *Practical.*
- *Half and half.*

We can now approximate level densities of “large” β by a sum of Gaussians!



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Applications:

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Outline.

1. A Central Limit Theorem (Kurt Johansson, 1998)

2. A (partial) combinatorial approach to the CLT based on the matrix models. (Alan Edelman and I.D., 2006)

3. Extension to a fuller result by concentration methods (Ofer Zeitouni and I.D., in preparation)

- **3.a.** Reduction to compactly supported functions (Gershgorin).
- **3.b.** Proof for a slightly different model (Guionnet-Zeitouni).
- **3.c.** Tying up the loose ends (Lidskii, Gershgorin).

A beautiful theorem.

CLT Theorem. (Johansson, 1998)
Let h be a "nice" function, and λ_i, $i = \overline{1,n}$ the *scaled* eigenvalues of the β-Hermite ensemble. Then

$$\sum_{i=1}^{n} h(\lambda_i) - \frac{2n}{\pi}\int_{-1}^{1} h(x)\sqrt{1-x^2}\mathrm{d}x \quad \Longrightarrow$$

$$\Longrightarrow \quad N\left(\left(1-\frac{2}{\beta}\right)\mu(h), \frac{2}{\beta}\sigma^2(h)\right) \ .$$

Remark 1. This is stronger than the semicircle law.
Remark 2. The theorem is even stronger; Hermite means potential $V(x) = x^2$; in fact, it works for $V(x)$ in a much larger polynomial class.

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$$\Longrightarrow \quad N\left(\left(1-\frac{2}{\beta}\right)\mu(h), \frac{2}{\beta}\,\sigma^2(h)\right) \; .$$

Works for

- **all** β,
- **many** $V(x)$,
- h **differentiable**, roughly **of polynomial growth** and with **polynomially-growing derivative**.

A combinatorial, matrix-based approach.

(Much) weaker CLT. (Edelman and D., 2006)
Let h be a polynomial, and λ_i, $i = \overline{1,n}$ the *scaled* eigenvalues of the β-Hermite ensemble. Then

$$\sum_{i=1}^{n} h(\lambda_i) - \frac{2n}{\pi}\int_{-1}^{1} h(x)\sqrt{1-x^2}\mathrm{d}x \quad \Longrightarrow$$

$$\Longrightarrow \quad N\left(\left(1-\frac{2}{\beta}\right)\mu(h), \frac{2}{\beta}\,\sigma^2(h)\right) \;.$$

Works for

- **all** β,
- $V(x) = x^2$,
- h **polynomial**.

Extension via linear algebra and probability.

(Slightly) stronger CLT for Hermite. (Zeitouni and D., -)
Let h be a "nice" function, and λ_i, $i = \overline{1, n}$ the *scaled* eigenvalues of the β-Hermite ensemble. Then

$$\sum_{i=1}^{n} h(\lambda_i) - \frac{2n}{\pi} \int_{-1}^{1} h(x) \sqrt{1 - x^2} \mathrm{d}x \quad \Longrightarrow$$

$$\Longrightarrow \quad N \left(\left(1 - \frac{2}{\beta} \right) \mu(h), \frac{2}{\beta}\, \sigma^2(h) \right) \; .$$

Works for

- **all** β,
- $V(x) = x^2$,
- h **Lipschitz** on open intervals, and **of polynomial growth**.

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Sketch of idea.

Would like to approximate a " nice" function h by a sequence of polynomials p_n (on compact sets); remainder should be small with high probability.

Let $p_n \to h$ on $C \in \mathbb{R}$, $n||p_n - h|| \to 0$ (plus some other regularity). Then

$$\int_{-1}^{1} p_n(x)\sqrt{1-x^2}\mathrm{d}x \to \int_{-1}^{1} h(x)\sqrt{1-x^2}\mathrm{d}x \ ;$$

Need to show (e.g.) that for any C,

$$\Pr\left[\left|\ \sum_{i=1}^{n} h(\lambda_i) - \sum_{i=1}^{n} p_n(\lambda_i)\ \right| \ > \ C\right] \ \leq \ \frac{c}{C^2}\,||p_n - h||\ .$$

Reduction to compactly supported functions.

We are essentially concerned with $\sum_{i=1}^{n} h(\lambda_i)$; we know (semicircle law) that

$$\frac{1}{n}\sum_{i=1}^{n}\delta_{\lambda_i} \to \mu_s \ ,$$

where $d\mu_s(x) = \frac{2}{\pi}\sqrt{1-x^2}$.

Enough to show that the probability of at least one of the eigenvalues being large (outside $[-K, K]$, $K \gg 1$) decays *exponentially* in K and n (since h will grow *polynomially*).

To show this we use Gershgorin's theorem.

Reduction to compactly supported functions.

Recall the *scaled* matrix

$$H_\beta \sim \frac{1}{2\sqrt{n\beta}} \begin{pmatrix} N(0,2) & \chi_{(n-1)\beta} & & & \\ \chi_{(n-1)\beta} & N(0,2) & \chi_{(n-2)\beta} & & \\ & \ddots & \ddots & \ddots & \\ & & \chi_{2\beta} & N(0,2) & \chi_\beta \\ & & & \chi_\beta & N(0,2) \end{pmatrix}$$

All Gershgorin disks have centers distributed like $N(0,1)/\sqrt{n}$, typical Gershgorin radius is distributed like $c\,\chi_{\alpha n}/\sqrt{n}$, $0 < \alpha < 1$.

But, for $K > c$,

$$\Pr\left[(|\,N(0,1)\,| \;+\; c\,\chi_{\alpha n}) \;>\; K\sqrt{n}\right] \leq c_1 e^{-c_2 K^2 n} ,$$

q.e.d.

A very useful result.

Theorem. (Guionnet-Zeitouni, 2000)
Let $A = (a_{ij})$ be a random matrix such that a_{ij} have distributions ν which satisfy the log-Sobolev inequality with uniform constant c, i.e., for all differentiable f,

$$\int f^2 \log \frac{f^2}{\int f^2 \mathrm{d}\nu} \mathrm{d}\nu \leq 2c \int |f'|^2 \mathrm{d}\nu \ ,$$

then for any Lipschitz function h, for any C,

$$\Pr\left[\left|\sum_{i=1}^{n} h(\lambda_i) - E\left[\sum_{i=1}^{n} h(\lambda_i)\right]\right| > C\right] \leq 2e^{-\frac{c_1}{h_{\mathcal{L}}}C^2} \ ,$$

where $h_{\mathcal{L}}$ is the Lipschitz constant for h.

Sounds perfect!

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except...

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the χ function does not satisfy log-Sobolev constraints.

Sounds perfect!

except...

the χ function does not satisfy log-Sobolev constraints.

But the Gaussian does.

Fixup: approximate χ by Gaussian:

$$\chi_r \sim c_r \ + \ \frac{1}{\sqrt{2}} N(0,1) \ + \quad small \ .$$

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A slightly perturbed model.

The Guionnet-Zeitouni theorem works for

$$\tilde{H}_\beta \sim \frac{1}{2\sqrt{n\beta}} \begin{pmatrix} N(0,2) & N(c_{n-1},\frac{1}{2}) & & & \\ N(c_{(n-1)},\frac{1}{2}) & N(0,2) & N(c_{(n-2)},\frac{1}{2}) & & \\ & \ddots & \ddots & \ddots & \\ & & N(c_2,\frac{1}{2}) & N(0,2) & N(c_1,\frac{1}{2}) \\ & & & N(c_1,\frac{1}{2}) & N(0,2) \end{pmatrix}$$

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Tying up loose ends.

Note that $H_{\text{diff}} = H_\beta - \tilde{H}_\beta$ looks, entry by entry, like

$$H_{\text{diff}} = O\left(\begin{pmatrix} 0 & \frac{1}{\sqrt{n(n-1)}} & & & \\ \frac{1}{\sqrt{n(n-1)}} & 0 & \frac{1}{\sqrt{n(n-2)}} & & \\ & \ddots & \ddots & \ddots & \\ & & \frac{1}{\sqrt{2n}} & 0 & \frac{1}{\sqrt{n}} \\ & & & \frac{1}{\sqrt{n}} & 0 \end{pmatrix}\right)$$

and the variables have uniformly bounded variance.

Tying up loose ends.

If λ_i are eigenvalues of H_β and $\tilde{\lambda}_i$ are eigenvalues of $\tilde{H}_\beta$, then for a Lipshitz h (with $h_{\mathcal{L}}$)

$$\left|\sum_{i=1}^{n} h(\lambda_i) - \sum_{i=1}^{n} h(\tilde{\lambda}_i)\right| \;\leq\; h_{\mathcal{L}} \sum_{i=1}^{n} |\lambda_i - \tilde{\lambda}_i| \;.$$

Tying up loose ends.

If λ_i are eigenvalues of H_β and $\tilde{\lambda}_i$ are eigenvalues of $\tilde{H}_\beta$, then for a Lipshitz h (with $h_{\mathcal{L}}$)

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By **Lidskii's theorem**,

$$\sum_{i=1}^{n} |\lambda_i - \tilde{\lambda}_i| \leq \sum_{i=1}^{n} |\lambda_{\mathrm{diff}_i}| \;,$$

where $\lambda_{\mathrm{diff}_i}$ are the eigenvalues of H_{diff}.

Tying up loose ends.

By **Gershgorin**,

$$\sum_{i=1}^{n} |\lambda_i - \tilde{\lambda}_i| \leq \sum_{i=1}^{n} O\left(\frac{1}{\sqrt{n}\sqrt{i}}\right) \ ,$$

and thus by **Chebyshev**,

$$\Pr\left[\left|\sum_{i=1}^{n} h(\lambda_i) - \sum_{i=1}^{n} h(\tilde{\lambda}_i)\right| \geq C\right] \ \leq \ h_{\mathcal{L}} \frac{c}{C^2} \ .$$

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