Integrable Geodesic Flows on Manifolds

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- Toda
- Symmetric rigid body equations – smooth and discrete
- Flows on Stiefel manfolds – Jacobi flow on ellipsoid
- Symmetric/Symplectic Flows and their Lie Poisson Structure
- Inspiration from Percy Deift.

Rigid Body Equations:

$$\dot{M} = [M, \Omega], \qquad M = \Lambda\Omega + \Omega\Lambda$$

Symmetric Rigid Body Equations:

$$\dot{Q} = Q\Omega \qquad \dot{P} = P\Omega$$

Toda Flow:

$$\dot{X} = [X, \Pi_S X]$$

Double Bracket Flow:

$$\dot{X} = [X, [X, N]]$$

– gradient but special case yields Toda.

Flow on the symmetric matrices/symplectic groups

$$\dot{X} = [X^2, N]$$

Matrix form of nonperiodic tridiagonal Toda:

$$\frac{d}{dt}L = [B, L] = BL - LB, \tag{0.1}$$

$$L = \begin{pmatrix} b_1 & a_1 & 0 & \cdots & 0 \\ a_1 & b_2 & a_2 & \cdots & 0 \\ & & \ddots & & \\ & & & b_{n-1} & a_{n-1} \\ 0 & & & a_{n-1} & b_n \end{pmatrix}$$

$$B = \begin{pmatrix} 0 & a_1 & 0 & \cdots & 0 \\ -a_1 & 0 & a_2 & \cdots & 0 \\ & & \ddots & & \\ & & & 0 & a_{n-1} \\ 0 & & & -a_{n-1} & 0 \end{pmatrix}$$

The double bracket flow is a gradient flow on an adjoint orbit $\mathcal{O}$ endowed with the "standard" or "normal" metric:

• When the matrix L in the double bracket flow is tridiagonal and the matrix N is the diagonal matrix $\mathrm{diag}(1, 2, \ldots n)$, the double bracket flow is both gradient and Hamiltonian on a level set of its integrals – the Toda lattice flow.

• Level set noncompact and diffeomorphic to a product of lines, unlike many Hamiltonian systems where the level set of the integrals is diffeomorphic to a torus.

Early key work on this: Moser, Symes, Deift, Nanda and Tomei.

Related flows: full Toda flows:

$$\dot{L} = [L, \pi_S L]$$

.

See work of Deift, Li, Nanda, Tomei; Ercolani, Flaschka, Singer.

1 The n-dimensional Rigid Body.

• Here review the classical rigid body equations in in n dimensions.

Use the following pairing on $\mathfrak{so}(n)$, the Lie algebra of the n-dimensional proper rotation group $\mathrm{SO}(n)$:

$$\langle \xi, \eta \rangle = -\frac{1}{2}\,\mathrm{trace}(\xi\eta).$$

Use this inner product to identify $\mathfrak{so}(n)^* \ \mathfrak{so}(n)$.

• Recall from Manakov [1976] and Ratiu [1980] that the left invariant generalized rigid body equations on $\mathrm{SO}(n)$ may be written as

$$\begin{aligned} \dot{Q} &= Q\Omega \\ \dot{M} &= [M, \Omega]\,, \end{aligned} \tag{RBn}$$

where $Q \in \mathrm{SO}(n)$ denotes the configuration space variable (the attitude of the body), $\Omega = Q^{-1}\dot{Q} \in \mathfrak{so}(n)$ is the body angular velocity, and the body angular momentum is

$$M := J(\Omega) = \Lambda\Omega + \Omega\Lambda \in \mathfrak{so}(n)\,.$$

- Here $J : \mathfrak{so}(n) \to \mathfrak{so}(n)$ is the symmetric pos def operator defined by

$$J(\Omega) = \Lambda\Omega + \Omega\Lambda,$$

where Λ is a diagonal matrix sat $\Lambda_i + \Lambda_j > 0$ for all $i \neq j$.

There is a similar formalism for any semisimple Lie group.

2 The Symmetric Rigid Body Equations.

The System (SRBn). By definition, ***the left invariant symmetric rigid body system*** (SRBn) is given by the first order equations

$$\begin{aligned} \dot{Q} &= Q\Omega \\ \dot{P} &= P\Omega \end{aligned} \tag{SRBn}$$

where Ω is regarded as a function of Q and P via the equations

$$\Omega := J^{-1}(M) \in \mathfrak{so}(n) \quad \text{and} \quad M := Q^T P - P^T Q.$$

Proposition 2.1. *If (Q, P) is a solution of (SRBn), then (Q, M) where $M = J(\Omega)$ and $\Omega = Q^{-1}\dot{Q}$ satisfies the rigid body equations (RBn).*

Proof. Differentiating $M = Q^T P - P^T Q$ and using the equations (SRBn) gives the second of the equations (RBn). ■

Proposition 2.2. *For a solution of the left invariant rigid body equations (RBn) obtained by means of Proposition 2.1, the spatial angular momentum is given by $m = PQ^T - QP^T$ and hence m is conserved along the rigid body flow.*

• Local Equivalence of the Rigid Body and the Symmetric Rigid Body Equations.

Above saw that solutions of the symmetric rigid body system can be mapped to solutions of the rigid body system. Now consider the converse question:

Suppose have a solution (Q, M) of the standard left invariant rigid body equations. Sseek to solve for P in

$$M = Q^T P - P^T Q. \tag{2.1}$$

Definition 2.3. *Let C denote the set of (Q, P) that map to M's with operator norm equal to 2 and let S denote the set of (Q, P) that map to M's with operator norm strictly less than 2. Also denote by S_M the set of points $(Q, M) \in T^* \operatorname{SO}(n)$ with $\|M\|_{\mathrm{op}} \leq 2$.*

Proposition 2.4. *For $\|M\|_{\mathrm{op}} < 2$, the equation(2.1) has the solution*

$$P = Q\left(e^{\sinh^{-1} M/2}\right) \tag{2.2}$$

The Hamiltonian Form of (SRBn).

Recall that the classical rigid body equations are Hamiltonian on $T^* \mathrm{SO}(n)$ with respect to the canonical symplectic structure on the cotangent bundle of $\mathrm{SO}(n)$.

In symmetric case have:

Proposition 2.5. *Consider the Hamiltonian system on the symplectic vector space* $\mathfrak{gl}(n) \times \mathfrak{gl}(n)$ *with the symplectic structure*

$$\Omega_{\mathfrak{gl}(n)}(\xi_1, \eta_1, \xi_2, \eta_2) = \frac{1}{2} \operatorname{trace}(\eta_2^T \xi_1 - \eta_1^T \xi_2) \tag{2.3}$$

and Hamiltonian

$$H(\xi, \eta) = -\frac{1}{8} \operatorname{trace}\left[\left(J^{-1}(\xi^T \eta - \eta^T \xi) \right) \left(\xi^T \eta - \eta^T \xi \right) \right] . \tag{2.4}$$

The corresponding Hamiltonian system leaves $\mathrm{SO}(n) \times \mathrm{SO}(n)$ *invariant and induces on it, the symmetric rigid body flow.*

Note that the above Hamiltonian is equivalent to

$$H = \frac{1}{4} \left\langle J^{-1} M, M \right\rangle .$$

3 Discrete Variational Problems

This general method is closely related to the development of variational integrators for the integration of mechanical systems, as in Kane, Marsden, Ortiz and West [2000]. See also Iserles, McLachlan, and Zanna [1999] and Budd and Iserles [1999].

Key notion: ***discrete Lagrangian***, which is a map $L_d : Q \times Q \to \mathbb{R}$. The important point here is that the velocity phase space TQ of Lagrangian mechanics has been replaced by $Q \times Q$.

In the discrete setting, the action integral of Lagrangian mechanics is replaced by an action sum

$$S_d = \sum_{k=0}^{N-1} L_d(q_k, q_{k+1}) \tag{3.1}$$

where $q_k \in Q$, the sum is over discrete time, and the equations are obtained by a discrete action principle which minimizes the discrete action given fixed endpoints q_0 and q_N.

Taking the extremum over $q_1, \cdots, q_{N-1}$ gives the ***discrete Euler-Lagrange equations***

$$D_2 L_d(q_{k-1}, q_k) + D_1 L_d(q_k, q_{k+1}) = 0, \tag{3.2}$$

for $k = 1, \cdots, N-1$.

We can rewrite this as follows

$$D_2 L_d + D_1 L_d \circ \Phi = 0, \tag{3.3}$$

where $\Phi : Q \times Q \to Q \times Q$ is defined implicitly by $\Phi(q_{k-1}, q_k) = (q_k, q_{k+1})$.

4 Moser–Veselov Discretization

Recall now the Moser–Veselov [1991] discrete rigid body equations. This system will be called DRBn.

See also Deift, Li and Tomei [1992].

Discretize the configuration matrix and let $Q_k \in \mathrm{SO}(n)$ denote the rigid body configuration at time k, let $\Omega_k \in \mathrm{SO}(n)$ denote the discrete rigid body angular velocity at time k, let I denote the diagonal moment of inertia matrix, and let M_k denote the rigid body angular momentum at time k.

These quantities are related by the Moser-Veselov equations

$$\Omega_k = Q_k^T Q_{k-1} \tag{4.1}$$

$$M_k = \Omega_k^T \Lambda - \Lambda \Omega_k \tag{4.2}$$

$$M_{k+1} = \Omega_k M_k \Omega_k^T . \tag{4.3}$$

(DRBn)

The Moser-Veslov equations (4.1)-(4.3) can in fact be obtained by a discrete variational principle (see Moser and Veselov [1991]) of the form described above: one considers the stationary points of the functional

$$S = \sum_k \operatorname{trace}(Q_k I Q_{k+1}) \tag{4.4}$$

on sequences of orthogonal $n \times n$ matrices.

See also Marsden, Pekarsky and Shkoller [1999].

The Discrete Symmetric Rigid Body.

We now define the symmetric discrete rigid body equations as follows:

$$
\begin{aligned}
Q_{k+1} &= Q_k U_k \\
P_{k+1} &= P_k U_k \,,
\end{aligned}
\tag{SDRBn}
$$

where U_k is defined by

$$
U_k \Lambda - \Lambda U_k^T = Q_k^T P_k - P_k^T Q_k \,. \tag{4.5}
$$

Using these equations, we have the algorithm $(Q_k, P_k) \mapsto (Q_{k+1}, P_{k+1})$ defined by: compute U_k from (4.5), compute Q_{k+1} and P_{k+1} using (SDRBn). We note that the update map for Q and P is done in parallel here.

Have:

Proposition 4.1. *The symmetric discrete rigid body equations (SDRBn) on S are equivalent to the Moser-Veselov equations (4.1)– (4.3) (DRBn) on the set S_M where S and S_M are defined in Proposition 2.3.*

Note that $m_k = P_k Q_k^T - Q_k P_k^T$ then $m_k = Q_k M_k Q_k^T$ and is conserved spatial momentum.

5 The Symmetric Rigid Body Equations with Parameter

Remarkable fact that the dynamic rigid body equations on $\mathrm{SO}(n)$ and indeed on any semisimple Lie group are integrable (Mishchenko and Fomenko [1976]).

• Key observation in this regard, due to Manakov, can write the generalized rigid body equations as Lax equations with parameter:

$$\frac{d}{dt}(M + \lambda\Lambda^2) = [M + \lambda\Lambda^2, \Omega + \lambda\Lambda], \tag{5.1}$$

where

$$M = J(\Omega) = \Lambda\Omega + \Omega\Lambda \tag{5.2}$$

as in Section 2.

Coefficients of λ in the traces of the powers of $M + \lambda\Lambda^2$ then yield the right number of independent integrals in involution to prove integrability of the flow on a generic adjoint orbit of $\mathrm{SO}(n)$ (identified with the corresponding coadjoint orbit).

• Moser and Veselov [1991] show that there is a corresponding formulation of the discrete rigid body equations with parameter.

In fact the full rigid body equations (kinematics plus dynamics) can be shown to be noncommutatively integrable (Mischenko and Fomenko [1978]).

Possible in fact also to write the full symmetric rigid body equations with parameter:

$$\begin{aligned} \dot{Q}_\lambda &= Q_\lambda(\Omega + \lambda\Lambda) \\ \dot{P}_\lambda &= P_\lambda(\Omega + \lambda\Lambda) \end{aligned} \tag{5.3}$$

where

$$\Omega = J^{-1}\left(M_\lambda - \lambda\Lambda^2\right) \tag{5.4}$$

and

$$M_\lambda = Q_\lambda^{-1}P_\lambda - P_\lambda^{-1}Q_\lambda. \tag{5.5}$$

Now we have

Lemma 5.1. *M_λ satisfies the rigid body equatons with parameter*

$$\dot{M}_\lambda = \left[M_\lambda, \Omega + \lambda\Lambda\right], \tag{5.6}$$

where $\Omega = J^{-1}\left(M_\lambda - \lambda\Lambda^2\right)$.

Note also that in the definition of M_λ here we use inverses rather than tranpose. This is crucial when we generalize from the pure finite group setting.

Integrals. We remark also that this leads to the Manakov integrals by the following argument: comparing (5.1) and (5.6) we see that $M_\lambda = J\Omega + \lambda\Omega^2$ is the solution of (5.6) by the same formal argument justifying the original Manakov equation.

6 Stiefel Manifolds

We introduce the variational and optimal control problems on a Stiefel manifold based on minimizing the time integral of the kinetic energy.

The metric on the manifold is given by the kinetic energy expression. We also give the extremal flows obtained in the limiting cases of the sphere/ellipsoid ($n = 1$), and the N dimensional rigid body ($n = N$). The extremal flows in these cases are well-known and integrable.

Variational Problems on a Stiefel Manifold

The Stiefel manifold $V(n, N) \subset \mathbb{R}^{nN}$ consists of orthogonal n frames in N dimensional real Euclidean space,

$$V(n, N) = \{Q \in \mathbb{R}^{nN}; \quad QQ^{\mathrm{T}} = I_n\}.$$

Introduce the pairing in $\mathbb{R}^{rs}$ given by

$$\langle A, B \rangle = \mathrm{Tr}(A^{\mathrm{T}} B), \tag{6.1}$$

where $\mathrm{Tr}(\cdot)$ denotes trace of a matrix and the left invariant metric on $\mathbb{R}^{nN}$ given by

$$\langle\langle W_1, W_2 \rangle\rangle = \langle W_1 \Lambda, W_2 \rangle = \langle W_1, W_2 \Lambda \rangle, \tag{6.2}$$

where Λ is a positive definite $N \times N$ diagonal matrix.

Consider the variational problem given by:

$$\min_{Q(\cdot)} \int_0^T \frac{1}{2} \langle\langle \dot{Q}, \dot{Q} \rangle\rangle dt \tag{6.3}$$

subject to: $QQ^{\mathrm{T}} = I_n$, $Q \in \mathbb{R}^{nN}$, $1 \leq n \leq N$, $Q(0) = Q_0$, $Q(T) = Q_T$, I_n denotes the $n \times n$ identity matrix. This is a variational problem defined on the Stiefel manifold $V(n, N)$. The dimension of this manifold is given by

$$\mathrm{Dim}\ V(n, N) = nN - \frac{n(n+1)}{2} = n(N-n) + \frac{n(n-1)}{2}.$$

Or:

$$\min_{U(\cdot)} \int_0^T \frac{1}{2} \langle\langle QU, QU \rangle\rangle dt \tag{6.4}$$

subject to: $\dot{Q} = QU$; $QQ^{\mathrm{T}} = I_n$, $Q(0) = Q_0$, $Q(T) = Q_T$ where $U \in \mathfrak{so}(N)$. Note that the quantity to be minimized is invariant with respect to the left action of $SO(n)$ on $V(n, N)$ since the metric (6.2) is left invariant.

The Rigid Body equations

For the special case when $n = N$, $V(N, N) \equiv SO(N)$ and the extremal trajectories of the optimal control problem (6.4) give the N-dimensional rigid body equations.

Geodesic flow on the ellipsoid

For the other extreme case, when $n = 1$, we obtain the equations for the geodesic flow on the sphere $V(1, N) \equiv \mathbb{S}^{N-1}$ with $Q = q^{\mathrm{T}}$, $q^{\mathrm{T}}q = 1$. This can be also be regarded as the geodesic flow on the ellipsoid

$$\bar{q}^{\mathrm{T}}\Lambda^{-1}\bar{q} = 1,$$

where $q = \Lambda^{-1/2}\bar{q}$. The costate variable $P = p^{\mathrm{T}}$ is used to enforce the constraint $\dot{q} = -Uq$ for the (6.4) when $n = 1$. The extremal solutions to this problem are

$$\dot{q} = -Uq, \quad \dot{p} = -Up + Aq, \tag{6.5}$$

where $A = qq^{\mathrm{T}}U\Lambda U - U\Lambda Uqq^{\mathrm{T}}$. The body momentum is obtained as

$$M = qp^{\mathrm{T}} - pq^{\mathrm{T}}, \tag{6.6}$$

in terms of the solution (q, p). Equations (6.5 can than be expressed in terms of the body momentum as

$$\dot{q} = -Uq, \quad \dot{M} = [M, U] - A. \tag{6.7}$$

The Lagrangian (variational) formulation for this problem gives us the equations for the geodesic flow on the sphere. To obtain these equations, we take reduced variations (see Marsden and Ratiu, 1999) on $V(1, N) = \mathbb{S}^{N-1}$. The equation of motion can be written as

$$\Lambda \ddot{q} = bq, \tag{6.8}$$

where b is a real scalar in this case. We get the Lagrangian (variational) equations for the geodesic flow on the sphere ($\mathbb{S}^{N-1}$) as

$$\ddot{q} = -\frac{\dot{q}^{\mathrm{T}}\dot{q}}{q^{\mathrm{T}}\Lambda^{-1}q}\Lambda^{-1}q. \tag{6.9}$$

Integrability of these extremal flows were proven by Jacobi with relation to Neumann problem of motion on sphere with quadratic potential, as shown by Knorrer (1982). Contemporary version of integrability of the geodesic flow on an ellipsoid was demonstrated by Moser (1980) using Theorem of Chasles and geometry of quadrics.

We can similarly obtain the equations for the general Stiefel case.
Obtain a symmetric form and discretization.

7 Flows on Symmetric Matrices and the Symplectic Group

Consider here analysis of the set of ordinary differential equations

$$\dot{X} = [X^2, N], \tag{7.1}$$

where $X \in \mathrm{Sym}(n)$, the linear space of $n \times n$ symmetric matrices, $\dot{X}$ denotes the time derivative, $N \in \mathfrak{so}(n)$, the space of skew symmetric $n \times n$ matrices, is given, and where initial conditions $X(0) = X_0 \in \mathrm{Sym}(n)$ are also given.

It is easy to check that $[X^2, N] \in \mathrm{Sym}(n)$, so that if the initial condition is in $\mathrm{Sym}(n)$, then $X(t) \in \mathrm{Sym}(n)$ for all t.

Also, because of the straightforward identity $\left[X^2, N\right] = [X, XN + NX]$, this equation may be rewritten in the Lax form

$$\dot{X} = [X, XN + NX], \tag{7.2}$$

again with initial conditions $X(0) = X_0 \in \mathrm{Sym}(n)$.

8 The Lie Algebra

We can regard N as a Poisson tensor on $\mathbb{R}^n$ by defining the bracket of two functions f, g as

$$\{f, g\}_N = (\nabla f)^T N \nabla g. \tag{8.1}$$

The Hamiltonian vector field associated with a function h (with the convention that $\dot{f}(z) = X_h(z) \cdot \nabla f(z) = \{f, h\}(z)$) is given by

$$X_h(z) = N \nabla h(z), \tag{8.2}$$

as is easily checked.

For each $X \in \mathrm{Sym}(n)$ define the quadratic Hamiltonian Q_X by

$$Q_X(z) := \frac{1}{2} z^T X z, \quad z \in \mathbb{R}^n.$$

Let $\mathcal{Q} := \{Q_X \mid X \in \mathrm{Sym}(n)\}$ be the vector space of all such functions. Note that the map $Q : X \in \mathrm{Sym}(n) \mapsto Q_X \in \mathcal{Q}$ is an isomorphism.

Using (8.2) it follows that the Hamiltonian vector field of Q_X has the form

$$X_{Q_X}(z) = N X z. \tag{8.3}$$

Next, we compute the Poisson bracket of two such quadratic functions.

Lemma 8.1. *For* $X, Y \in \mathrm{Sym}(n)$*, we have*

$$\{Q_X, Q_Y\}_N = Q_{[X,Y]_N}, \tag{8.4}$$

where $[X,Y]_N = XNY - YNX \in \mathrm{Sym}(n)$*. In addition,* $\mathrm{Sym}(n)$ *is a Lie algebra relative to the Lie bracket* $[\cdot,\cdot]_N$*. Therefore,* $Q : X \in (\mathrm{Sym}(n), [\cdot,\cdot]_N) \mapsto Q_X \in (\mathcal{Q}, \{\cdot,\cdot\}_N)$ *is a Lie algebra isomorphism.*

It is a general fact that Hamiltonian vector fields and Poisson brackets are related by

$$[X_f, X_g] = -X_{\{f,g\}}, \tag{8.5}$$

where the bracket on the left hand side is the Jacobi-Lie bracket. Thus, it is natural to look at the corresponding algebra of Hamiltonian vector fields on the Poisson manifold $(\mathbb{R}^n, \{\cdot,\cdot\}_N)$ associated to quadratic Hamiltonians. If we take $f = Q_X$ and $g = Q_Y$, with $X_f = NX$ and $X_g = NY$, and recall that the Jacobi-Lie bracket of *linear* vector fields is the negative of the commutator of the associated matrices, then we have

Proposition 8.2. *Equations (8.4) and (8.5) imply*

$$N[X,Y]_N = [NX, NY]\,. \tag{8.6}$$

We can of course one verify this by hand. Letting $\mathcal{LH}$ denote the Lie algebra of linear Hamiltonian vector fields on $\mathbb{R}^n$ relative to the commutator bracket of matrices, (8.6) states that the map

$$X \in (\mathrm{Sym}(n), [\cdot,\cdot]_N) \mapsto NX \in (\mathcal{LH}, [\cdot,\cdot])$$

is a homomorphism of Lie algebras.

Invertible Case.

If N is invertible, then this homomorphism is an isomorphism. In addition, the non-degeneracy of N implies that n is even and that $\mathbb{R}^n$ is a symplectic vector space relative to the symplectic form defined by N^{-1}. Therefore, the Lie algebra $(\mathcal{LH}, [\cdot,\cdot])$ is isomorphic to the Lie algebra $\mathfrak{sp}(\mathbb{R}^n, N^{-1})$ of linear symplectic maps of $\mathbb{R}^n$ relative to the symplectic form N^{-1}, that is to the classical Lie algebra $\mathfrak{sp}(n, \mathbb{R})$.

We summarize these considerations in the following statement.

Proposition 8.3. *Let $N \in \mathfrak{so}(n)$. The map $Q : X \in (\mathrm{Sym}(n), [\cdot,\cdot]_N) \mapsto Q_X \in (\mathcal{Q}, \{\cdot,\cdot\}_N)$ is a Lie algebra isomorphism. The map $X \in (\mathrm{Sym}(n), [\cdot,\cdot]_N) \mapsto NX \in (\mathcal{LH}, [\cdot,\cdot])$ is a Lie algebra homomorphism and if N is invertible it induces an isomorphism of $(\mathrm{Sym}(n), [\cdot,\cdot]_N)$ with $\mathfrak{sp}(n, \mathbb{R})$.*

Euler-Poincaré Form

Identify $\mathrm{Sym}(n)$ with its dual using the the positive definite inner product

$$\langle\!\langle X, Y \rangle\!\rangle := \mathrm{trace}\,(XY)\,, \quad \text{for} \quad X, Y \in \mathrm{Sym}(n). \tag{8.7}$$

Remark. The inner product $\langle\!\langle X, Y \rangle\!\rangle$ is not ad invariant relative to the N-bracket, but another one, namely $\kappa_N(X, Y) := \mathrm{trace}(NXNY)$ is invariant, as is easy to check.

Define the Lagrangian $l : \mathrm{Sym}(n) \to \mathbb{R}$ on the Lie algebra $(\mathrm{Sym}(n), [\cdot, \cdot]_N)$ by

$$l(X) = \frac{1}{2}\,\mathrm{trace}\left(X^2\right) = \frac{1}{2}\,\mathrm{trace}\left(XX^T\right) =: \frac{1}{2}\,\langle\!\langle X, X \rangle\!\rangle\,. \tag{8.8}$$

Proposition 8.4. *The equations*

$$\dot{X} = [X^2, N] \tag{8.9}$$

are the Euler-Poincaré equations corresponding to the Lagrangian (8.8) *on the Lie algebra* $(\mathrm{Sym}(n), [\cdot, \cdot]_N)$.

Noninvertible case Let $2p = \operatorname{rank} N$ and $d := n - 2p$. Then $\bar{N} := N|_{\operatorname{im} N} : \operatorname{im} N \to \operatorname{im} N$ defines a nondegenerate skew symmetric bilinear form and, by the previous proposition, $(\mathrm{Sym}(2p), [\cdot, \cdot]_{\bar{N}})$ is isomorphic as a Lie algebra to $(\mathfrak{sp}(\mathbb{R}^{2p}, \bar{N}^{-1}), [\cdot, \cdot])$.

Proposition 8.5. *Can find a map*

$$\Psi : ((\mathrm{Sym}(2p) \circledS \mathcal{M}_{(2p)\times d}) \oplus \mathrm{Sym}(d), [\cdot, \cdot]^C) \to (\mathrm{Sym}(n), [\cdot, \cdot]_N)$$

given by

$$\Psi(S, A, B) := \begin{bmatrix} S & A \\ A^T & B \end{bmatrix} \tag{8.10}$$

which is a Lie algebra isomorphism.

Poisson structure

Identifying Sym(n) with its dual using the inner product (8.7) endows Sym(n) with the the (left, or minus) Lie Poisson bracket

$$\{f, g\}_N(X) = -\operatorname{trace}\Big[X\Big(\nabla f(X)N\nabla g(X) - \nabla g(X)N\nabla f(X)\Big)\Big], \tag{8.11}$$

where ∇f is the gradient of f relative to the inner product $\langle\langle\cdot,\cdot\rangle\rangle$ on Sym(n). It is easy to check that the equations $\dot{X} = \left[X^2, N\right]$ are Hamiltonian relative to the function l defined in (8.8) and the Lie-Poisson bracket (8.11).

Later on we shall also need the *frozen* Poisson bracket

$$\{f, g\}_{FN}(X) = -\operatorname{trace}\Big(\nabla f(X)N\nabla g(X) - \nabla g(X)N\nabla f(X)\Big). \tag{8.12}$$

It is a general fact that the Poisson structures (8.11) and (8.12) are *compatible* in the sense that their sum is a Poisson structure.

Proposition 8.6. *Let* $n = 2p + d$*, where* $2p = \operatorname{rank} N$*. The generic leaves of the Lie-Poisson bracket* $\{\cdot, \cdot\}_N$ *are* $2p(p + d)$*-dimensional.*

Proposition 8.7. *All leaves of the frozen Poisson bracket* $\{\cdot, \cdot\}_{FN}$ *are*

(i) $2p(p+d)$*-dimensional if* N *is generic, that is, all its non-zero eigenvalues are distinct, and*

(ii) $p(p + 1 + 2d)$*-dimensional if all non-zero eigenvalue pairs of* N *are equal.*

For what follows it is important to compute the Poisson tensors corresponding to the above Poisson brackets. Recall that the Poisson tensor can be viewed as a vector bundle morphism $B : T^*(\mathrm{Sym}(n)) \to T(\mathrm{Sym}(n))$ covering the identity. It is defined by $B(\mathbf{d}h) = \{\cdot, h\}_N$ for any locally defined smooth function h on $\mathrm{Sym}(n)$. Since $\mathrm{Sym}(n)$ is a vector space, these bundles are trivial and hence the value B_X at $X \in \mathrm{Sym}(n)$ of the Poisson tensor B is a linear map $B_X : \mathrm{Sym}(n) \to \mathrm{Sym}(n)$ by identifying $\mathrm{Sym}(n)$ with its dual using the inner product $\langle\langle \cdot, \cdot \rangle\rangle$.

Proposition 8.8. *Denote the value at $X \in \mathrm{Sym}(n)$ of the Poisson tensors corresponding to the Lie-Poisson (8.11) and frozen (8.12) brackets by B_X and C_X, respectively. Then for any $Y \in \mathrm{Sym}(n)$ we have*

$$B_X(Y) = XYN - NYX \tag{8.13}$$

$$C_X(Y) = YN - NY. \tag{8.14}$$

Proposition 8.9 (Casimir Functions). *Assume that N is invertible so that n is even, say $n = 2p$.*

(i) *If N is the standard skew matrix $\mathbb{J}$, then the Casimir functions for the frozen Poisson structure* (8.12) *are of the form*

$$C_F^k(X) = \mathrm{trace}(S_k X), \quad k = 1, \dots p,$$

(ii) *The Casimir functions for the Lie-Poisson bracket $\{\cdot, \cdot\}_N$ are given by*

$$C^k(X) = \frac{1}{2k} \,\mathrm{trace}\left[\left(XN^{-1}\right)^{2k}\right], \quad \textit{for} \quad k = 1, \dots, p.$$

Note: Can find similar bur more complicated formulae for the Casimirs in the noninvertible case.

Note: can check: system is not of Mischenko-Fomenko sectional operator type.

9 Lax Pairs with Parameter

To prove that system (7.1) is integrable for any choice of N, we will compute its flow invariants. Bear it in mind that, by virtue of the isospectral representation (7.2), we already know that the eigenvalues of X, or alternatively, the quantities trace X^k for $k = 1, 2, \ldots, n-1$, are invariants.

One way to compute additional invariants is to rewrite the system as a Lax pair with a parameter. One can do this in a fashion similar to that for the generalized rigid body equations (see Manakov).

Theorem 9.1. *Let λ be a real parameter. The system (7.2) is equivalent to the following Lax pair system*

$$\frac{d}{dt}(X + \lambda N) = \left[X + \lambda N, NX + XN + \lambda N^2\right] \tag{9.1}$$

Recall Manakov:

$$\frac{d}{dt}(M + \lambda \Lambda^2) = [M + \lambda \Lambda^2, \Omega + \lambda \Lambda]. \tag{9.2}$$

For the generalized rigid body the nontrivial coefficients of $\lambda^i, 0 < i < k$ in the traces of the powers of $M + \lambda\Lambda^2$ then yield the right number of independent integrals in involution to prove integrability of the flow on a generic adjoint orbit of $SO(n)$ (identified with the corresponding coadjoint orbit). The case $i = 0$ needs to be eliminated, because these are Casimir functions.

Similarly, in our case, the nontrivial coefficients of $\lambda^i, 0 \leq i \leq k$, in

$$h_k^\lambda(X) := \frac{1}{k}\operatorname{trace}(X + \lambda N)^k, \qquad k = 1, 2, \ldots, n-1 \tag{9.3}$$

yield the conserved quantities. The coefficient of $\lambda^r, 0 \leq r \leq k$, in (9.3) is

$$\operatorname{trace} \sum_{|i|=k-r} \sum_{|j|=r} X^{i_1} N^{j_1} X^{i_2} \cdots X^{i_s} N^{j_s}, \qquad r = 1, \ldots, k-1, \quad k = 1, \ldots, n-1,$$

where $i = (i_1, i_2, \ldots i_s)$, $j = (j_1, j_2, \ldots j_s)$ are multi-indices, $i_q, j_q = 0, 1, \ldots, k-1$, and $|i| = \sum_{q=1}^s i_q$, $|j| = \sum_{q=1}^s j_q$. The coefficient of λ^k is the constant N^k so it should not be counted. Thus we have $r < k$. In addition, since the trace of a matrix equals the trace of its transpose, $X \in \operatorname{Sym}(n)$, and $N \in \mathfrak{so}(n)$, it follows that

$$\operatorname{trace} X^{i_1} N^{j_1} X^{i_2} \cdots X^{i_s} N^{j_s} = (-1)^{|j|} \operatorname{trace} N^{j_s} X^{j_s} \cdots X^{i_2} N^{j_1} X^{i_1}.$$

Thus, we are left with the nontrivial invariants

$$\operatorname{trace} \sum_{|i|=k-2r} \sum_{|j|=2r} X^{i_1} N^{j_1} X^{i_2} \cdots X^{i_s} N^{j_s} \tag{9.4}$$

for $i_q, j_q = 0, \ldots, k-1$, $r = 1, \ldots, \left[\frac{k-1}{2}\right]$, where $[p]$ denotes the integer part of $p \in \mathbb{R}$. Altogether, this results in

$$\left[\frac{n}{2}\right]\left[\frac{n+1}{2}\right]$$

invariants as an easy inductive argument shows.

Are these integrals the right candidates to prove complete integrability of the system $\dot{X} = [X^2, N]$?

- If N is invertible, then $n = 2p$ and hence

$$\begin{aligned}\left[\frac{n}{2}\right]\left[\frac{n+1}{2}\right] &= \left[\frac{2p}{2}\right]\left[\frac{2p+1}{2}\right] = p^2 = \frac{1}{2}\left(2p^2 + p - p\right) \\ &= \frac{1}{2}\left(\dim \mathfrak{sp}(2p, \mathbb{R}) - \operatorname{rank} \mathfrak{sp}(2p, \mathbb{R})\right)\end{aligned}$$

which is half the dimension of the generic adjoint orbit in $\mathfrak{sp}(2p, \mathbb{R})$. Therefore, these conserved quantities are the right candidates to prove that this system is integrable on the generic coadjoint orbit of $\mathrm{Sym}(n)$.

- If N is non-invertible (which is equivalent to $d \neq 0$), then $n = 2p + d$ and hence

$$\begin{aligned}\left[\frac{n}{2}\right]\left[\frac{n+1}{2}\right] &= \left[\frac{2p+d}{2}\right]\left[\frac{2p+d+1}{2}\right]\\ &= \left(p + \left[\frac{d}{2}\right]\right)\left(p + \left[\frac{d+1}{2}\right]\right)\\ &= p^2 + p\left(\left[\frac{d}{2}\right] + \left[\frac{d+1}{2}\right]\right) + \left[\frac{d}{2}\right]\left[\frac{d+1}{2}\right]\\ &= p^2 + pd + \left[\frac{d}{2}\right]\left[\frac{d+1}{2}\right].\end{aligned}$$

The right number of integrals is $p(p+d)$ according to Proposition 8.6, so this calculation seems to indicate that there are additional integrals. The situation is not so simple since there are redundancies due to the degeneracy of N. Note, however, that if $d = 1$, then we do get the right number of integrals.

10 Integrability

This section shows that the Hamiltonian system (7.1) is integrable in the case $n = 2p$.

Bihamiltonian structure. We begin with the following observation.

Proposition 10.1. *The system $\dot{X} = X^2N - NX^2$ is Hamiltonian with respect to the bracket $\{f,g\}_N$ defined in (8.11) using the Hamiltonian $h_2(X) := \frac{1}{2}\,\mathrm{trace}(X^2)$ and is also Hamiltonian with respect to the compatible bracket $\{f,g\}_{FN}$ defined in (8.12) using the Hamiltonian $h_3(X) := \frac{1}{3}\,\mathrm{trace}(X^3)$.*

Involution. We prove that the $\left[\frac{n}{2}\right]\left[\frac{n+1}{2}\right]$ integrals given in (9.4), namely

$$h_{k,2r}(X) := \mathrm{trace} \sum_{|i|=k-2r} \sum_{|j|=2r} X^{i_1}N^{j_1}X^{i_2}\cdots X^{i_s}N^{j_s},$$

where $i_q, j_q = 0,\ldots,k-1$, $r = 1,\ldots,\left[\frac{k-1}{2}\right]$, $k = 1,\ldots,n-1$, are in involution. Denote by $h_{k,k-r}$ the coefficient of λ^{k-r} in $\frac{1}{k}\,\mathrm{trace}\,(X+\lambda N)^k$ so that we have

$$h_k^\lambda(X) = \frac{1}{k}\,\mathrm{trace}\,(X+\lambda N)^k = \sum_{r=0}^{k} \lambda^{k-r} h_{k,k-r}(X)\,. \tag{10.1}$$

As explained before, not all of these coefficients should be counted: roughly half of them vanish and the last one, namely, $h_{k,k}$, is the constant N^k. Consistent with our notation for the Hamiltonians, we set $h_k = h_{k,0}$.

Firstly we need the gradients of the functions h_k^λ.

Lemma 10.2. *The gradients* ∇h_k^λ *are given by*

$$\nabla h_k^\lambda(X) = \frac{1}{2}(X + \lambda N)^{k-1} + \frac{1}{2}(X - \lambda N)^{k-1}. \tag{10.2}$$

Proposition 10.3.

$$B_X(\nabla h_k^\lambda(X)) = C_X(\nabla h_{k+1}^\lambda(X)) \tag{10.3}$$

Proposition 10.4. *The functions* $h_{k,k-r}$ *satisfy the recursion relation*

$$B_X(\nabla h_{k,k-r}(X)) = C_X(\nabla h_{k+1,k-r}(X)) \tag{10.4}$$

Remark. It is worth making a few remarks about Propositions 10.3 and 10.4. Note that unlike the similar recursion for the rigid body Mankov integrals our polynomial recursion relation (10.3) does not have a premultiplier λ on the right hand side and the polynomials on the left and right hand sides appear to be of different order. This cannot be and indeed is not so. Indeed the highest order order coefficient on the right hand side vanishes by virtue of following result.

Corollary 10.5. *The functions $h_{k,k-1}(X)$ are Casimirs for the frozen Poisson structure, i.e.*

$$C_X\left(\nabla h_{k,k-1}(X)\right) = 0 \tag{10.5}$$

for all k.

Proof. By (10.1), $h_{k,k-1}(X) = \operatorname{trace}\left(N^{k-1}X\right)$, so its gradient equals $\nabla h_{k,k-1}(X) = N^{k-1}$. So (8.14) immediately gives (10.5). ■

The recursion relations (10.4) for $r = 0$ also imply the following relation between the Hamiltonians that can also be easily checked by hand.

Corollary 10.6.

$$B_X\left(\nabla h_k(X)\right) = C_X\left(\nabla h_{k+1}(X)\right) \tag{10.6}$$

Example: An interesting nontrivial example of the recursion relation to check is $B_X(dh_{3,2}(X)) = C_X(dh_{4,2}(X))$ where $h_{3,2}(X) = \mathrm{trace}(N^2X)$ and $h_{4,2}(X) = \mathrm{trace}(N^2X^2) + \frac{1}{2}\mathrm{trace}(NXNX)$. This example illustrates how the recursion relation works despite the apparent inconsistency in order.

Using the recursion relations involution follows immediately.

Proposition 10.7. *The invariants $h_{k,k-r}$ are in involution with respect to both Poisson brackets $\{f,g\}_N$ and $\{f,g\}_{FN}$.*

Proof. The definition of the Poisson tensors B_X and C_X and the recursion relation (10.4) give

$$\begin{aligned}\{h_{k,k-r}, h_{l,l-q}\}_N &= \langle\langle \nabla h_{k,k-r}(X), B_X(\nabla h_{l+1,l-q}(X))\rangle\rangle \\ &= \langle\langle \nabla h_{k,k-r}(X), C_X(\nabla h_{l+1,l-q}(X))\rangle\rangle \\ &= \{h_{k,k-r}, h_{l+1,l-q}\}_{FN} = -\{h_{l+1,l-q}, h_{k,k-r}\}_{FN} \\ &= -\langle\langle \nabla h_{l+1,l-q}(X), C_X(\nabla h_{k,k-r}(X))\rangle\rangle \\ &= -\langle\langle \nabla h_{l+1,l-q}(X), B_X(\nabla h_{k-1,k-r}(X))\rangle\rangle \\ &= -\{h_{l+1,l-q}, h_{k-1,k-r}\}_N = \{h_{k-1,k-r}, h_{l+1,l-q}\}_N\end{aligned}$$

for any $k, l = 1, \ldots, n-1$, $r = 1, \ldots, k$ and $q = 0, \ldots, l-1$.Of course, in these relations we assume that $k-r$ and $l-q$ are even, for if at least one of them is odd, the identity above has zeros on both sides. Repeated application of this relation eventually leads to Hamiltonians $h_{k,k-r}$ where either $k-r$ is a power that does not exist for k, in which case the Hamiltonian is zero, or one is led to $h_{0,0}$ which is constant. This shows that $\{h_{k,k-r}, h_{l,l-q}\}_N = 0$ for any pair of indices.

In a similar way one shows that $\{h_{k,k-r}, h_{l,l-q}\}_{FN} = 0$. ■

Independence

Theorem 10.8. *For generic N the integrals $h_{k,2r}$ given by equation (9.4) are independent.*

Hence, since we have involution and independence we have proved the following.

Theorem 10.9. *For N invertible with distinct eigenvalues the system* (7.1) *is completely integrable.*

Corollary 10.10. *For N odd with distinct eigenvalues and nullity one, the system* (7.1) *is completely integrable.*

It is also of interest to analyze linearization on the Jacobi variety of the curve

$$\det(zI - \lambda N - X) = 0$$

– use work of Adler/van Moerbeke. Griffiths.