# Integrable Systems: Painlevé–Chazy–Ramanujan

Mark J. Ablowitz
Department of Applied Mathematics
University of Colorado, Boulder

# Outline

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- Reductions of self-dual Yang-Mills (SDYM) and a 3x3 system related to Darboux-Halphen system– “DH-9”:
- Solution of DH-9 via Schwarzian Equations
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- Classical Chazy eq.–another form of solution: Modular forms
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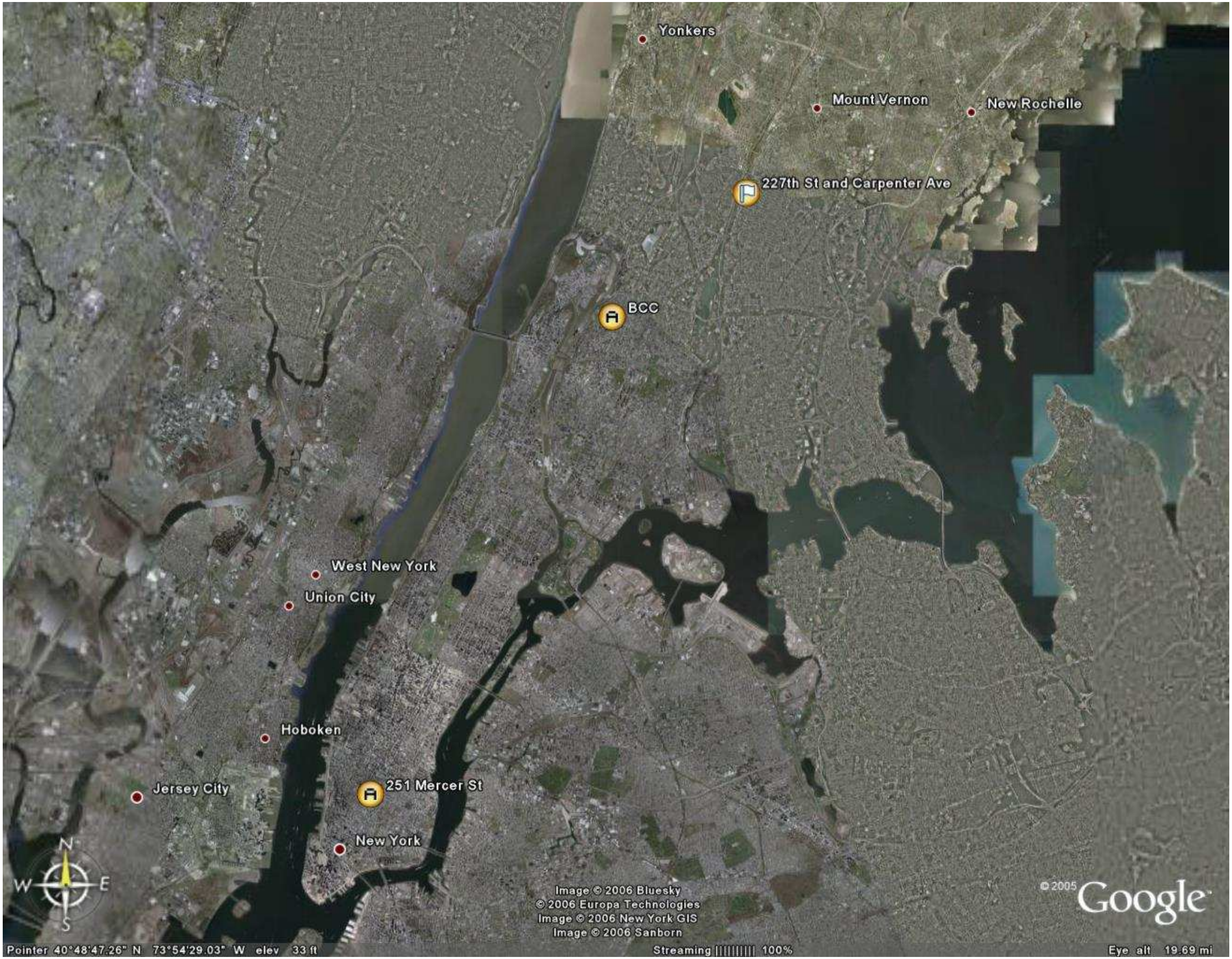
- Nevalinna theory–discrete Painlevé type equations
- Conclusion



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# Introduction

Wide interest in integrable systems; many mathematically and physically interesting systems.

$1+1$ dimension

- KdV: $u_t + 6uu_x + u_{xxx} = 0$
- mKdV: $u_t \pm 6u^2 u_x + u_{xxx} = 0$
- NLS: $iu_t + u_{xx} \pm 2|u|^2 u = 0$

$2+1$ dimension

- KP: $(u_t + 6uu_x + u_{xxx})_x \pm 3u_{yy} = 0$
- DS: $iu_t + u_{xx} + \sigma_1 u_{yy} + \phi u = 0$

$$\phi_{xx} - \sigma_1 \phi_{yy} = 2\sigma_2 (|u|^2)_{xx} \; ; \quad \sigma_j = \pm 1; j = 1, 2$$

# Solutions

- Rapid decay:
  Riemann-Hilbert BVP; DBAR $\Rightarrow$
  Linear integral equations
  Soliton solutions
  2+1dim: rapid decay from line solitons
- Periodic/quasi-periodic solutions
  Systems of ODE’s
  transform to multidimensional theta functions
- Self-similar solutions
  ODE-Painlevé type
- Automorphic functions: Darboux-Halphen-Chazy class

# Self-similar solutions

$$u_t + 6uu_x + u_{xxx} = 0$$

1973: MJA & A. Newell: from Gel'fand-Levitan-Marchenko
$t \to \infty \Rightarrow$ self-similar solution

$u(x,t) \sim \frac{1}{(3t)^{2/3}} f(z), \quad z = \frac{x}{(3t)^{1/3}}$ for $|\frac{x}{(3t)^{1/3}}| = O(1)$

$$f''' + 6ff' - (zf' + 2f) = 0$$

Note: $f = -(w' + w^2)$ leads to second Painlevé equation:

$$w'' - (zw + 2w^3) = \alpha, \quad PII$$

$\alpha =$const. Similarity Solution's are physically important.

# Asymptotics mKdV: $t \to \infty$

$$u_t - 6u^2 u_x + u_{xxx} = 0$$

Asymptotic analysis leads to a *slowly varying similarity solution* (cf. MJA & H. Segur, 1977-1981; MJA: Courant lecture: Feb. 1978)

$$u(x,t) \sim \frac{1}{(3t)^{1/3}} w(z), \quad z = \frac{x}{(3t)^{1/3}}$$

$$w'' - (zw + 2w^3) = 0$$

$w = w(z, c_1, c_2)$, where $c_i = c_i(\xi)$, $\xi = x/t$.

From slowly varying similarity solution: when $\xi = x/t \to 0$ => connection formulae for PII

# Connection Formulae– PII

$$w'' - (zw + 2w^3) = 0$$

$w(z) \sim r_0 Ai(z), z \to \infty$

$w(z) \sim \frac{d_0}{|z|^{1/4}} sin\theta, z \to -\infty$

where: $\theta = \frac{2}{3}|z|^{3/2} - \frac{3}{2}d_0^2 log|z| + \theta_0; |r_0| < 1$

Find the *connection formulae*

$d_0 = -\frac{1}{\pi} log(1 - |r_0|^2)$

$\theta_0 = \frac{\pi}{4} - \frac{3log2}{2} d_0^2 - arg\{\Gamma(1 - i\frac{d_0^2}{2})\}$

Thus given the constant $r_0$ as $z \to \infty$ we have explicit formulae for the values of the constants as $z \to -\infty$, i.e.

$d_0 = d_0(r_0)$

$\theta_0 = \theta_0(r_0)$

Cf. (MJA & H. Segur, 1981)

# Asymptotics $t \to \infty$ (con't)

Many other workers have studied long time asymptotic solutions of integrable systems, notably S. Manakov and Deift, Zhou and co-workers who developed stationary phase methods for Riemann-Hilbert problems.

Another problem where such "modulated" similarity solutions arise is critical self focusing of Benney-Roskes-Davey-Stewartson coupled NLS-type systems. cf. MJA, I. Bakirtas and B. Ilan, 2005, and ref. therein.

Context and breadth slowly varying similarity solutions associated with asymptotic solutions is still open.

# Integrable systems–ODE's of P-Type

Self-similar reductions of integrable systems–MJA, A. Ramani, H. Segur: 1977,1978, 1980, 1981.

Reductions: mKdV => PII; Boussinesq => PI; Sine-Gordon => PIII;...; (SDYM) => all six Painlevé equations in general position; cf. Mason and Woodhouse, 1993, 1996

Painlevé (P) type equations have no movable branch points. NLPDE's solvable by inverse scattering transform (IST) deeply connected to P- type equations; the solutions of the underlying linear integral equations only yield movable poles

Solution of NLPDE via IST; solution of P-type equations IMT (iso-monodromy transform)
cf. Flashka & Newell '80. MJA & Fokas '83, Mason & Woodhouse '96, Bolibruch, Its, Kapaev '04...
Open: complete analysis of PVI via IMT

# P-Type Equations

- P-Type: ODE has no movable branch points

  Fuch's, Kovalaveskya, Painlevé, Gambier, Chazy...
- 1st order ODE:

$$y' = F(z, y)$$

  Rational in y, locally analytic (l.a.) in $z$
  Find: only Ricatti equation of P-Type:
  $\frac{dy}{dz} = a_0(z) + a_1(z)y + a_2(z)y^2$
- 2nd order ODE:

$$y'' = F(z, y, y')$$

  Rational in $y, y'$, l.a. in $z$. 50 classes of equations; including linear eq., reductions to Ricatti and and 6 Painlevé transcendents.

# Painlevé equations

$$y'' = 6y^2 + z, \quad PI$$

$$y'' = zy + y^3 + \alpha, \quad PII$$

$\alpha$=const.

$$y'' = \frac{y'^2}{y} - \frac{y'}{z} + \frac{\alpha y^2 + \beta}{z} + \gamma y^3 + \frac{\delta}{y}, \quad PIII$$

$\alpha, \beta, \gamma, \delta$=const.

...

Third order equations: full classification of $y''' = F(y, y', y'', z)$ still open. Chazy (1909-1911) found interesting systems with movable natural boundaries.

# Reduction SDYM

SDYM:

$$F_{\alpha\beta} = 0, F_{\bar{\alpha}\bar{\beta}} = 0$$

$$F_{\alpha\bar{\alpha}} + F_{\beta\bar{\beta}} = 0$$

where

$$F_{\alpha\beta} = \partial_\alpha \gamma_\beta - \partial_\beta \gamma_\alpha - [\gamma_\alpha, \gamma_\beta]$$

and $[\gamma_\alpha, \gamma_\beta] = \gamma_\alpha \gamma_\beta - \gamma_\beta \gamma_\alpha$

Cartesian cood.: $\alpha = t + iz, \bar{\alpha} = t - iz, \beta = x + iy, \bar{\beta} = x - iy$

Reductions of SDYM:

1. $\gamma_a(\alpha, \bar{\alpha}, \beta, \bar{\beta}) -> \gamma_a(\alpha), \gamma_a(\alpha, \beta), ...$
2. choice of algebra: $gl(N), su(N)...$
3. gauge freedom: $\gamma_a -> (f\gamma_a - \partial_a f)f^{-1}$

# 1D Reductions of SDYM

Take:

$\gamma_\alpha = \gamma_t + i\gamma_z = \gamma_0 + i\gamma_3$

$\gamma_\beta = \gamma_x + i\gamma_y = \gamma_1 + i\gamma_2$

guage: $\gamma_0 = 0$; $\gamma_j = \gamma_j(t), j = 1, 2, 3$

$$F_{\alpha\beta} = \partial_\alpha \gamma_\beta - \partial_\beta \gamma_\alpha - [\gamma_\alpha, \gamma_\beta] = \partial_t(\gamma_1 + i\gamma_2) - [i\gamma_3, \gamma_1 + i\gamma_2] = 0$$

Formally, real, imaginary parts =>

$$\partial_t \gamma_1 = [\gamma_2, \gamma_3], \quad 1, 2, 3 \ \ cyclic$$

Simplest case: $\gamma_l(t) = \omega_l(t) X_l$; $\ su(2): \ [X_j, X_k] = \sum_l \epsilon_{jkl} X_l$

where $\epsilon_{jkl}$ is antisym tensor and $\epsilon_{123} = 1$. Find

$$\partial_t \omega_1 = \omega_2 \omega_3, \quad 1, 2, 3 \ \ cyclic$$

# 1D Reductions of SDYM–con't

$$\partial_t \omega_1 = \omega_2 \omega_3, \quad 1,2,3 \ \ cyclic$$

Note:

$$\omega_1 = E cosh \phi(t), \ \omega_2 = E sinh \phi(t), \ \omega_3 = \frac{d\phi(t)}{dt}$$

$E$=const. find:

$$\frac{d^2\phi}{dt^2} = \frac{E^2}{2} sinh \phi$$

Solution is in terms of elliptic functions.

# Darboux-Halphen Systems

$$\partial_t \gamma_1 = [\gamma_2, \gamma_3], \quad 1,2,3 \ \ cyclic$$

Set $\gamma_l(t) = \sum_{j,k} O_{lj} M_{jk}(t) X_k$ where:

$[X_j, X_k] = \sum_l \epsilon_{jkl} X_l, \ OO^T = I, O\epsilon \ so(3),$
$X_l(O_{jk}) = \sum_p \epsilon_{lkp} O_{jp}, \ sdiff(S^3)$

Find $M = \{M_{jk}(t)\}$ satisfies:

$$\frac{dM}{dt} = (detM)(M^{-1})^T + M^T M - (TrM)M \quad (DH - 9)$$

Chakravarty, MJA, Takhtajan, 1992. $M = diag(\omega_1, \omega_2, \omega_3)$ find

$$\partial_t \omega_1 = \omega_2 \omega_3 - \omega_1(\omega_2 + \omega_3), \ \ 1,2,3 \ \ cyclic \ (DH)$$

Chakravarty, MJA, Clarkson, 1990

# Solution of DH-9

Solution of DH-9 can be written in terms of Schwarzian functions (MJA,Chakravarty, Halburd, 1999) which satisfy

$$\{s,t\} + \frac{\dot{s}^2}{2} V(s) = 0$$

where $V(s) = \frac{1-\beta^2}{s^2} + \frac{1-\gamma^2}{(s-1)^2} + \frac{\beta^2+\gamma^2-\alpha^2-1}{s(s-1)}$. $s(t)$ conformal map $t \to s$; $s(t)$ single valued if

$$\alpha = \frac{1}{l}, \beta = \frac{1}{m}, \gamma = \frac{1}{n}, \quad l, m, n \, \epsilon \, Z$$

Schwarzian eq. can be linearized: $\{s,t\} = -\dot{s}^2\{t,s\}$,

i.e. $t(s) = y_1/y_2$ where $y_j, j = 1,2$ satisfy: $y'' + \frac{1}{4}V(s)y = 0.$
Solution has a movable natural boundary which is a circle. Radius and center depend on I.C.'s.

# Solution of DH-9-details

$$\frac{dM}{dt} = (detM)(M^{-1})^T + M^T M - (TrM)M \quad (DH-9)$$
$$M = P(D+a)P^{-1}$$

where $P, D, a$ satisfy:

$$\frac{dP}{dt} = -Pa, \quad D = diag(\omega_1, \omega_2, \omega_3), \quad a_{ij} = \sum_k \epsilon_{ijk}\tau_k$$

$$\partial_t \omega_1 = \omega_2\omega_3 - \omega_1(\omega_2 + \omega_3) + \tau^2, \quad 1,2,3 \ \ cyclic$$

$$\tau^2 = \sum_k \tau_k^2, \quad \partial_t \tau_1 = -\tau_1(\omega_2 + \omega_3), \quad 1,2,3 \ \ cyclic$$

# Solution of DH-9–details con't

$$\omega_1 = -\frac{1}{2}\frac{d}{dt}log\frac{\dot{s}}{s(s-1)}, \quad \omega_2 = -\frac{1}{2}\frac{d}{dt}log\frac{\dot{s}}{s-1}, \quad \omega_3 = -\frac{1}{2}\frac{d}{dt}log\frac{\dot{s}}{s}$$

$$\tau_1 = \frac{\kappa_1\dot{s}}{[s(s-1)]^{1/2}}, \quad \tau_2 = \frac{\kappa_2\dot{s}}{s(s-1)^{1/2}}, \quad \tau_3 = \frac{\kappa_3\dot{s}}{s^{1/2}(s-1)}$$

$\kappa_j = const., \ j = 1, 2, 3$ where $s(t)$ satisfies a Schwarzian eq.

$$\{s, t\} + \frac{\dot{s}^2}{2}V(s) = 0$$

where

$V(s) = \frac{1-\beta^2}{s^2} + \frac{1-\gamma^2}{(s-1)^2} + \frac{\beta^2+\gamma^2-\alpha^2-1}{s(s-1)}; \alpha = -2\kappa_1^2, \beta = 2\kappa_2^2, \gamma = -2\kappa_3^2$

# Solution of DH-9–Schwarzian

$$\{s,t\} + \frac{\dot{s}^2}{2} V(s) = 0$$

where

$$V(s) = \frac{1-\beta^2}{s^2} + \frac{1-\gamma^2}{(s-1)^2} + \frac{\beta^2+\gamma^2-\alpha^2-1}{s(s-1)}$$

$s(t)$ conformal map $t \to s$; $s(t)$ single valued if

$$\alpha = \frac{1}{l}, \beta = \frac{1}{m}, \gamma = \frac{1}{n}, \quad l, m, n \epsilon Z$$

Schwarzian eq. can be linearized: $\{s,t\} = -\dot{s}^2\{t,s\}$, $t(s) = y_1/y_2$ where $y_j, j = 1,2$ satisfy: $y'' + \frac{1}{4}V(s)y = 0$. Solution has a movable natural boundary which is a circle. Radius and center depend on initial conditions.

# Darboux-Halphen -Chazy

When $M = diag(\omega_1, \omega_2, \omega_3)$ DH-9 reduces to:

$$\partial_t \omega_1 = \omega_2\omega_3 - \omega_1(\omega_2 + \omega_3), \quad 1,2,3 \;\; cyclic \; (DH)$$

Let $y = -2(\omega_1 + \omega_2 + \omega_3)$ find classical Chazy eq.

$$\frac{d^3y}{dt^3} - 2y\frac{d^2y}{dt^2} + 3(\frac{dy}{dt})^2 = 0 \quad (C)$$

When: $\alpha = \beta = \gamma = 2/n$ (DH-9) yields for $y = -2TrM$

$$\frac{d^3y}{dt^3} - 2y\frac{d^2y}{dt^2} + 3(\frac{dy}{dt})^2 = \frac{4}{36 - n^2}(6\frac{dy}{dt} - y^2)^2 \;\; (GC)$$

GC: Gneralized Chazy eq. (MJA,Chakravarty, Halburd, 1999).
$n = \infty$ => (C). Chazy eq. have movable natural boundary–circle.

# Chazy Eq. – Modular Functions

$$\frac{d^3y}{dt^3} - 2y\frac{d^2y}{dt^2} + 3(\frac{dy}{dt})^2 = 0 \quad \text{(C)}$$

(C) admits symmetry:

$$y \longrightarrow \tilde{y} = \frac{1}{(ct+d)^2}y(\gamma(t)) - \frac{6c}{ct+d}, \quad \gamma(t) = \frac{at+b}{ct+d}$$

where $ad - bc = 1$. Special solution of (C)

$$y(t) = i\pi E_2(t) = i\pi(1 - 24\sum_{n=1}^{\infty}\sigma_1(n)q^n), \quad q = e^{2\pi it}$$

$\sigma_1(n) = \sum_{d/n} d$ = sum of divisors of $n$; $E_2(t)$ satisfies above symmetry with $a, b, c, d$ integers–it is a quasi-modular form. MJA, Chakravarty,Takhtajan 1991.

# Chazy Eq. – Modular Func.–con't

Solution of Chazy also written as: $y(t) = \frac{1}{2}\frac{d}{dt}log\Delta(t)$
where

$$\Delta(t) = \frac{\Delta(\gamma(t))}{(ct+d)^{12}} = Cq\prod_{1}^{\infty}(1-q^n)^{24} = \sum_{1}^{\infty}\tau(n)q^n$$

$\gamma(t) = \frac{at+b}{ct+d},\ q = e^{2\pi it}, C = (2\pi)^{12},\ \tau(n) =$ Ramanujan coef.

$\Delta(t)$ satisfies

$$\Delta_4\Delta^3 - 5\Delta_3\Delta_1\Delta^2 - \frac{3}{2}\Delta_2^2\Delta^2 + 12\Delta\Delta_1^2\Delta_2 - \frac{13}{2}\Delta_1^4 = 0$$

where $\Delta_p = \frac{d^p\Delta}{dt^p}$; Rankine: 1956.

# Chazy and Ramanujan Equations

Ramnujan (1916) showed that the arithmetic functions

$P(q) = (1 - 24\sum_{n=1}^{\infty} \sigma_1(n)q^n) = E_2(t)$
$Q(q) = (1 + 240\sum_{n=1}^{\infty} \sigma_3(n)q^n) = E_4(t)$
$R(q) = (1 - 504\sum_{n=1}^{\infty} \sigma_5(n)q^n) = E_6(t)$
$\sigma_k(n) = \sum_{d/n} d^k$=sum of divisors of $n$ to $k$th power, satisfy

$$\frac{dP}{dt} = \frac{i\pi}{6}(P^2 - Q) \qquad (1)$$
$$\frac{dQ}{dt} = \frac{2i\pi}{3}(PQ - R) \qquad (2)$$
$$\frac{dR}{dt} = i\pi(PR - Q^2) \qquad (3)$$

From (1): $Q = P^2 + \frac{6i}{\pi}\frac{dP}{dt}$; then (2) => $R = R[p, \frac{dP}{dt}, \frac{d^2P}{dt^2}]$

Eq. (3) is a 3rd order eq. for $P(t)$. Letting $P(t) = \frac{y(t)}{i\pi}$ =>

$$\frac{d^3y}{dt^3} - 2y\frac{d^2y}{dt^2} + 3(\frac{dy}{dt})^2 = 0 \quad Chazy => P, Q, R!$$

# Chazy and Ramanujan Eq. -con't

Ramanujan type system for generalized Chazy:

$$\frac{dP}{dt} = \frac{i\pi}{6}(P^2 - Q) \qquad (1')$$
$$\frac{dQ}{dt} = \frac{2i\pi}{3}(PQ - R) \qquad (2')$$
$$\frac{dR}{dt} = i\pi(PR - Q^2(1 + \frac{36}{n^2-36})) \qquad (3')$$

3rd order eq. for $P(t) = \frac{y(t)}{i\pi}$ =>

$$\frac{d^3y}{dt^3} - 2y\frac{d^2y}{dt^2} + 3(\frac{dy}{dt})^2 = \frac{4}{36 - n^2}(6\frac{dy}{dt} - y^2)^2 \quad (GC)$$

# Conclusion

- Reductions of integrable systems yield: Painlevé and Chazy type equations.
- In particular, reduction of SDYM => $3x3$ matrix system: DH-9. This system can be solved in terms of Schwarzian triangle functions.
- Special cases include Classical Chazy and Generalized Chazy eq.
- Classical Chazy also has solution $E_2(t)$
- Ramanujan found a 3rd order system for $E_j(t), \ j = 2, 4, 6$ which reduces to Classical Chazy (MJA,Chakravarty, Halburd, 2003)

# Conclusion–con't

Thus Chazy (1909-1911) and Ramanujan (1916) worked on the same equations . But from a totally different perspective. Hahn recently showed, using previous result of Ramamani, that functions in a subgroup of SL(2,Z): $\Gamma_0(N)$ satisfies a different 3rd order scalar eq.

Open:

- Obtain other integrable systems that of number theoretic importance.
- Solve DH-9 by moving monodromy methods; cf. Chakravarty, MJA 1996 for "DH-5".
- Are there interesting "1+1, 2+1" DH- type systems that can be studied by IST methods?