Integrable Systems:
Painlevé–Chazy–Ramanujan

Mark J. Ablowitz
Department of Applied Mathematics
University of Colorado, Boulder
Outline

Introduction

Painlevé equations–integrable systems

Reductions of self-dual Yang-Mills (SDYM) and a 3x3 system related to Darboux-Halphen system– “DH-9”:

Solution of DH-9 via Schwarzian Equations

Reductions of DH-9 to “generalized” and “classical” Chazy equations

Classical Chazy eq.–another form of solution: Modular forms

Connection to differential equations of Ramanujan

Nevalinna theory–discrete Painlevé type equations

Conclusion
Introduction

Wide interest in integrable systems; many mathematically and physically interesting systems.

1 + 1 dimension

- KdV: \[ u_t + 6uu_x + u_{xxx} = 0 \]
- mKdV: \[ u_t \pm 6u^2u_x + u_{xxx} = 0 \]
- NLS: \[ iu_t + u_{xx} \pm 2|u|^2u = 0 \]

2 + 1 dimension

- KP: \[ (u_t + 6uu_x + u_{xxx})_x \pm 3u_{yy} = 0 \]
- DS: \[ iu_t + u_{xx} + \sigma_1u_{yy} + \phi u = 0 \]
  \[ \phi_{xx} - \sigma_1\phi_{yy} = 2\sigma_2(|u|^2)_{xx} ; \quad \sigma_j = \pm 1; \quad j = 1, 2 \]
Solutions

- Rapid decay:
  Riemann-Hilbert BVP; DBAR ⇒
  Linear integral equations
  Soliton solutions
  2+1dim: rapid decay from line solitons

- Periodic/quasi-periodic solutions
  Systems of ODE’s
  transform to multidimensional theta functions

- Self-similar solutions
  ODE-Painlevé type

- Automorphic functions: Darboux-Halphen-Chazy class
Self-similar solutions

\[ u_t + 6uu_x + u_{xxx} = 0 \]

1973: MJA & A. Newell: from Gel’fand-Levitan-Marchenko
\[ t \to \infty \Rightarrow \text{self-similar solution} \]

\[ u(x,t) \sim \frac{1}{(3t)^{2/3}} f(z), \quad z = \frac{x}{(3t)^{1/3}} \text{ for } |\frac{x}{(3t)^{1/3}}| = O(1) \]

\[ f''' + 6f f' - (zf' + 2f) = 0 \]

Note: \( f = -(w' + w^2) \) leads to second Painlevé equation:

\[ w'' - (zw + 2w^3) = \alpha, \quad PII \]

\( \alpha = \text{const. Similarity Solution’s are physically important.} \)
Asymptotics mKdV: \( t \to \infty \)

\[ u_t - 6u^2u_x + u_{xxx} = 0 \]


\[ u(x, t) \sim \frac{1}{(3t)^{1/3}} w(z), \quad z = \frac{x}{(3t)^{1/3}} \]

\[ w'' - (zw + 2w^3) = 0 \]

\( w = w(z, c_1, c_2) \), where \( c_i = c_i(\xi), \quad \xi = x/t. \)

From slowly varying similarity solution: when \( \xi = x/t \to 0 \Rightarrow \) connection formulae for PII
Connection Formulae– PII

\[ w'' - (zw + 2w^3) = 0 \]

\[ w(z) \sim r_0Ai(z), z \to \infty \]

\[ w(z) \sim \frac{d_0}{|z|^{1/4}}sin\theta, z \to -\infty \]

where: \[ \theta = \frac{2}{3}|z|^{3/2} - \frac{3}{2}d_0^2log|z| + \theta_0; |r_0| < 1 \]

Find the connection formulae

\[ d_0 = -\frac{1}{\pi}log(1 - |r_0|^2) \]

\[ \theta_0 = \frac{\pi}{4} - \frac{3log^2d_0^2}{2} - arg\{\Gamma(1 - i\frac{d_0^2}{2})\} \]

Thus given the constant \( r_0 \) as \( z \to \infty \) we have explicit formulae for the values of the constants as \( z \to -\infty \), i.e.

\[ d_0 = d_0(r_0) \]

\[ \theta_0 = \theta_0(r_0) \]

Cf. (MJA & H. Segur, 1981)
Asymptotics $t \to \infty$ (con’t)

Many other workers have studied long time asymptotic solutions of integrable systems, notably S. Manakov and Deift, Zhou and co-workers who developed stationary phase methods for Riemann-Hilbert problems.

Another problem where such "modulated" similarity solutions arise is critical self focusing of Benney-Roskes-Davey-Stewartson coupled NLS-type systems. cf. MJA, I. Bakirtas and B. Ilan, 2005, and ref. therein.

Context and breadth slowly varying similarity solutions associated with asymptotic solutions is still open.
Integrable systems–ODE’s of P-Type


Reductions: mKdV => PII; Boussinesq => PI; Sine-Gordon => PIII;...; (SDYM) => all six Painlevé equations in general position; cf. Mason and Woodhouse, 1993, 1996

Painlevé (P) type equations have no movable branch points. NLPDE’s solvable by inverse scattering transform (IST) deeply connected to P- type equations; the solutions of the underlying linear integral equations only yield movable poles

Solution of NLPDE via IST; solution of P-type equations IMT (iso-monodromy transform) cf. Flashka & Newell ’80. MJA & Fokas ’83, Mason & Woodhouse ’96, Bolibruch, Its, Kapaev ’04...
Open: complete analysis of PVI via IMT
P-Type Equations

- P-Type: ODE has no movable branch points

  - Fuch’s, Kovalaveskya, Painlevé, Gambier, Chazy...

- 1st order ODE:

  \[ y' = F(z, y) \]

  Rational in \( y \), locally analytic (l.a.) in \( z \)

  Find: only Ricatti equation of P-Type:

  \[ \frac{dy}{dz} = a_0(z) + a_1(z)y + a_2(z)y^2 \]

- 2nd order ODE:

  \[ y'' = F(z, y, y') \]

  Rational in \( y, y' \), l.a. in \( z \). 50 classes of equations; including linear eq., reductions to Ricatti and and 6 Painlevé transcendents.
Painlevé equations

\[ y'' = 6y^2 + z, \quad PI \]

\[ y'' = zy + y^3 + \alpha, \quad PII \]

\( \alpha = \text{const.} \)

\[ y'' = \frac{y'^2}{y} - \frac{y'}{z} + \frac{\alpha y^2 + \beta}{z} + \gamma y^3 + \frac{\delta}{y}, \quad PIII \]

\( \alpha, \beta, \gamma, \delta = \text{const.} \)

... Third order equations: full classification of \( y''' = F(y, y', y'', z) \) still open. Chazy (1909-1911) found interesting systems with movable natural boundaries.
Reduction SDYM

SDYM:

\[ F_{\alpha\beta} = 0, F_{\bar{\alpha}\bar{\beta}} = 0 \]
\[ F_{\alpha\bar{\alpha}} + F_{\beta\bar{\beta}} = 0 \]

where

\[ F_{\alpha\beta} = \partial_\alpha \gamma_\beta - \partial_\beta \gamma_\alpha - [\gamma_\alpha, \gamma_\beta] \]

and

\[ [\gamma_\alpha, \gamma_\beta] = \gamma_\alpha \gamma_\beta - \gamma_\beta \gamma_\alpha \]

Cartesian coord.: \( \alpha = t + iz, \bar{\alpha} = t - iz, \beta = x + iy, \bar{\beta} = x - iy \)

Reductions of SDYM:

1. \( \gamma_a(\alpha, \bar{\alpha}, \beta, \bar{\beta}) \rightarrow \gamma_a(\alpha), \gamma_a(\alpha, \beta), \ldots \)
2. choice of algebra: \( gl(N), su(N) \ldots \)
3. gauge freedom: \( \gamma_a \rightarrow (f \gamma_a - \partial_a f)f^{-1} \)
1D Reductions of SDYM

Take:
\( \gamma_\alpha = \gamma_t + i\gamma_z = \gamma_0 + i\gamma_3 \)
\( \gamma_\beta = \gamma_x + i\gamma_y = \gamma_1 + i\gamma_2 \)

\[ \text{guage: } \gamma_0 = 0; \gamma_j = \gamma_j(t), j = 1, 2, 3 \]

\[ F_{\alpha\beta} = \partial_\alpha \gamma_\beta - \partial_\beta \gamma_\alpha - [\gamma_\alpha, \gamma_\beta] = \partial_t (\gamma_1 + i\gamma_2) - [i\gamma_3, \gamma_1 + i\gamma_2] = 0 \]

Formally, real, imaginary parts =>

\[ \partial_t \gamma_1 = [\gamma_2, \gamma_3], \quad 1, 2, 3 \quad \text{cyclic} \]

Simplest case: \( \gamma_1(t) = \omega_1(t)X_1; \quad \text{su}(2): [X_j, X_k] = \sum_l \epsilon_{jkl}X_l \)
where \( \epsilon_{jkl} \) is antisym tensor and \( \epsilon_{123} = 1 \). Find

\[ \partial_t \omega_1 = \omega_2 \omega_3, \quad 1, 2, 3 \quad \text{cyclic} \]
1D Reductions of SDYM–con’t

\[ \partial_t \omega_1 = \omega_2 \omega_3, \quad 1, 2, 3 \text{ cyclic} \]

Note:

\[ \omega_1 = E \cosh \phi(t), \quad \omega_2 = E \sinh \phi(t), \quad \omega_3 = \frac{d \phi(t)}{dt} \]

\( E = \text{const. find:} \)

\[ \frac{d^2 \phi}{dt^2} = \frac{E^2}{2} \sinh \phi \]

Solution is in terms of elliptic functions.
Darboux-Halphen Systems

\[ \partial_t \gamma_1 = [\gamma_2, \gamma_3], \quad 1, 2, 3 \text{ cyclic} \]

Set \( \gamma_l(t) = \sum_{j,k} O_{lj} M_{jk}(t) X_k \) where:

\[ [X_j, X_k] = \sum_l \epsilon_{jkl} X_l, \quad OO^T = I, \quad O \epsilon \text{ so}(3), \]
\[ X_l(O_{jk}) = \sum_p \epsilon_{lpk} O_{jp}, \quad \text{diff}(S^3) \]

Find \( M = \{M_{jk}(t)\} \) satisfies:

\[ \frac{dM}{dt} = (\det M)(M^{-1})^T + M^T M - (\text{Tr} M) M \quad (DH - 9) \]

Chakravarty, MJA, Takhtajan, 1992. \( M = \text{diag}(\omega_1, \omega_2, \omega_3) \) find

\[ \partial_t \omega_1 = \omega_2 \omega_3 - \omega_1(\omega_2 + \omega_3), \quad 1, 2, 3 \text{ cyclic (DH)} \]

Chakravarty, MJA, Clarkson, 1990
Solution of DH-9

Solution of DH-9 can be written in terms of Schwarzian functions (MJA, Chakravarty, Halburd, 1999) which satisfy

\[ \{s, t\} + \frac{s'^2}{2} V(s) = 0 \]

where \( V(s) = \frac{1-\beta^2}{s^2} + \frac{1-\gamma^2}{(s-1)^2} + \frac{\beta^2+\gamma^2-\alpha^2-1}{s(s-1)} \). \( s(t) \) conformal map \( t \to s; \) \( s(t) \) single valued if

\[ \alpha = \frac{1}{l}, \beta = \frac{1}{m}, \gamma = \frac{1}{n}, \quad l, m, n \in \mathbb{Z} \]

Schwarzian eq. can be linearized: \( \{s, t\} = -s^2 \{t, s\}, \)

i.e. \( t(s) = \frac{y_1}{y_2} \) where \( y_j, j = 1, 2 \) satisfy:

\[ y'' + \frac{1}{4} V(s)y = 0. \]

Solution has a movable natural boundary which is a circle. Radius and center depend on I.C.’s.
Solution of DH-9-details

\[
\frac{dM}{dt} = (\text{det}M)(M^{-1})^T + M^T M - (\text{Tr}M)M \quad (DH - 9)
\]

\[
M = P(D + a)P^{-1}
\]

where \(P, D, a\) satisfy:

\[
\frac{dp}{dt} = -Pa, \quad D = \text{diag}(\omega_1, \omega_2, \omega_3), \quad a_{ij} = \sum_k \epsilon_{ijk} \tau_k
\]

\[
\partial_t \omega_1 = \omega_2 \omega_3 - \omega_1(\omega_2 + \omega_3) + \tau^2, \quad 1,2,3 \quad \text{cyclic}
\]

\[
\tau^2 = \sum_k \tau_k^2, \quad \partial_t \tau_1 = -\tau_1(\omega_2 + \omega_3), \quad 1,2,3 \quad \text{cyclic}
\]
Solution of DH-9–details con’t

\[ \omega_1 = -\frac{1}{2} \frac{d}{dt} \log \frac{\dot{s}}{s(s-1)}, \quad \omega_2 = -\frac{1}{2} \frac{d}{dt} \log \frac{\dot{s}}{s-1}, \quad \omega_3 = -\frac{1}{2} \frac{d}{dt} \log \frac{\dot{s}}{s} \]

\[ \tau_1 = \frac{\kappa_1 \dot{s}}{[s(s-1)]^{1/2}}, \quad \tau_2 = \frac{\kappa_2 \dot{s}}{s(s-1)^{1/2}}, \quad \tau_3 = \frac{\kappa_3 \dot{s}}{s^{1/2}(s-1)} \]

\[ \kappa_j = \text{const.}, \quad j = 1, 2, 3 \] where \( s(t) \) satisfies a Schwarzian eq.

\[ \{s, t\} + \frac{\dot{s}^2}{2} V(s) = 0 \]

where

\[ V(s) = \frac{1-\beta^2}{s^2} + \frac{1-\gamma^2}{(s-1)^2} + \frac{\beta^2 + \gamma^2 - \alpha^2 - 1}{s(s-1)}; \quad \alpha = -2\kappa_1^2, \beta = 2\kappa_2^2, \gamma = -2\kappa_3^2 \]
Solution of DH-9–Schwarzian

\[ \{s, t\} + \frac{s^2}{2} V(s) = 0 \]

where

\[ V(s) = \frac{1 - \beta^2}{s^2} + \frac{1 - \gamma^2}{(s - 1)^2} + \frac{\beta^2 + \gamma^2 - \alpha^2 - 1}{s(s - 1)} \]

s(t) conformal map \( t \rightarrow s \); s(t) single valued if

\[ \alpha = \frac{1}{l}, \beta = \frac{1}{m}, \gamma = \frac{1}{n}, \quad l, m, n \in \mathbb{Z} \]

Schwarzian eq. can be linearized: \( \{s, t\} = -\dot{s}^2 \{t, s\}, \quad t(s) = y_1/y_2 \)

where \( y_j, j = 1,2 \) satisfy: \( y'' + \frac{1}{4} V(s)y = 0 \). Solution has a movable natural boundary which is a circle. Radius and center depend on initial conditions.
When $M = \text{diag}(\omega_1, \omega_2, \omega_3)$ DH-9 reduces to:

$$\partial_t \omega_1 = \omega_2 \omega_3 - \omega_1(\omega_2 + \omega_3), \quad 1, 2, 3 \text{ cyclic (DH)}$$

Let $y = -2(\omega_1 + \omega_2 + \omega_3)$ find classical Chazy eq.

$$\frac{d^3 y}{dt^3} - 2y \frac{d^2 y}{dt^2} + 3\left(\frac{dy}{dt}\right)^2 = 0 \quad (C)$$

When: $\alpha = \beta = \gamma = 2/n$ (DH-9) yields for $y = -2TrM$

$$\frac{d^3 y}{dt^3} - 2y \frac{d^2 y}{dt^2} + 3\left(\frac{dy}{dt}\right)^2 = \frac{4}{36 - n^2} \left(6 \frac{dy}{dt} - y^2\right)^2 \quad (GC)$$

GC: Generalized Chazy eq. (MJA, Chakravarty, Halburd, 1999).

$n = \infty \Rightarrow \text{(C)}. \text{ Chazy eq. have movable natural boundary–circle.}$
Chazy Eq. – Modular Functions

\[
\frac{d^3 y}{dt^3} - 2y \frac{d^2 y}{dt^2} + 3\left(\frac{dy}{dt}\right)^2 = 0 \quad (C)
\]

(C) admits symmetry:

\[
y \rightarrow \tilde{y} = \frac{1}{(ct + d)^2} y(\gamma(t)) - \frac{6c}{ct + d}, \quad \gamma(t) = \frac{at + b}{ct + d}
\]

where \(ad - bc = 1\). Special solution of (C)

\[
y(t) = i\pi E_2(t) = i\pi(1 - 24\sum_{n=1}^{\infty} \sigma_1(n)q^n), \quad q = e^{2\pi it}
\]

\(\sigma_1(n) = \sum_{d|n} d = \text{sum of divisors of } n\); \(E_2(t)\) satisfies above symmetry with \(a, b, c, d\) integers—it is a quasi-modular form. MJA, Chakravarty, Takhtajan 1991.
Solution of Chazy also written as: 

\[ y(t) = \frac{1}{2} \frac{d}{dt} \log \Delta(t) \]

where

\[ \Delta(t) = \frac{\Delta(\gamma(t))}{(ct + d)^{12}} = Cq \prod_{1}^{\infty} (1 - q^n)^{24} = \sum_{1}^{\infty} \tau(n)q^n \]

\[ \gamma(t) = \frac{at+b}{ct+d}, \quad q = e^{2\pi it}, \quad C = (2\pi)^{12}, \quad \tau(n) = \text{Ramanujan coef.} \]

\( \Delta(t) \) satisfies

\[ \Delta_4 \Delta^3 - 5\Delta_3 \Delta_1 \Delta^2 - \frac{3}{2} \Delta_2^2 \Delta^2 + 12\Delta \Delta_1^2 \Delta_2 - \frac{13}{2} \Delta_1^4 = 0 \]

where \( \Delta_p = \frac{d^p \Delta}{dt^p} \); Rankine: 1956.
Chazy and Ramanujan Equations

Ramnujan (1916) showed that the arithmetic functions

\[ P(q) = (1 - 24 \sum_{n=1}^{\infty} \sigma_1(n)q^n) = E_2(t) \]
\[ Q(q) = (1 + 240 \sum_{n=1}^{\infty} \sigma_3(n)q^n) = E_4(t) \]
\[ R(q) = (1 - 504 \sum_{n=1}^{\infty} \sigma_5(n)q^n) = E_6(t) \]

\( \sigma_k(n) = \sum_{d|n} d^k \) = sum of divisors of \( n \) to \( k \)th power, satisfy

\[ \frac{dP}{dt} = \frac{i\pi}{6}(P^2 - Q) \quad (1) \]
\[ \frac{dQ}{dt} = \frac{2i\pi}{3}(PQ - R) \quad (2) \]
\[ \frac{dR}{dt} = i\pi(PR - Q^2) \quad (3) \]

From (1): \( Q = P^2 + \frac{6i}{\pi} \frac{dP}{dt} \); then (2) \( \Rightarrow R = R[p, \frac{dP}{dt}, \frac{d^2P}{dt^2}] \)

Eq. (3) is a 3rd order eq. for \( P(t) \). Letting \( P(t) = \frac{y(t)}{i\pi} \) \( \Rightarrow \)

\[ \frac{d^3y}{dt^3} - 2y\frac{d^2y}{dt^2} + 3\left(\frac{dy}{dt}\right)^2 = 0 \quad \text{Chazy} \Rightarrow P, Q, R! \]
Chazy and Ramanujan Eq. -con’t

Ramanujan type system for generalized Chazy:

\[
\frac{dP}{dt} = \frac{i\pi}{6}(P^2 - Q) \quad (1')
\]
\[
\frac{dQ}{dt} = \frac{2i\pi}{3}(PQ - R) \quad (2')
\]
\[
\frac{dR}{dt} = i\pi(PR - Q^2(1 + \frac{36}{n^2-36})) \quad (3')
\]

3rd order eq. for \( P(t) = \frac{y(t)}{i\pi} \) =>

\[
\frac{d^3y}{dt^3} - 2y\frac{d^2y}{dt^2} + 3\left(\frac{dy}{dt}\right)^2 = \frac{4}{36 - n^2}(6\frac{dy}{dt} - y^2)^2 \quad (GC)
\]
Conclusion

Reductions of integrable systems yield: Painlevé and Chazy type equations.

In particular, reduction of SDYM => $3 \times 3$ matrix system: DH-9. This system can be solved in terms of Schwarzian triangle functions.

Special cases include Classical Chazy and Generalized Chazy eq.

Classical Chazy also has solution $E_2(t)$

Ramanujan found a 3rd order system for $E_j(t)$, $j = 2, 4, 6$ which reduces to Classical Chazy (MJA, Chakravarty, Halburd, 2003)
Thus Chazy (1909-1911) and Ramanujan (1916) worked on the same equations. But from a totally different perspective. Hahn recently showed, using previous result of Ramamani, that functions in a subgroup of \( \text{SL}(2,\mathbb{Z}) : \Gamma_0(N) \) satisfies a different 3rd order scalar eq.

Open:

- Obtain other integrable systems that of number theoretic importance.
- Solve DH-9 by moving monodromy methods; cf. Chakravarty, MJA 1996 for “DH-5”.
- Are there interesting "1+1, 2+1" DH-type systems that can be studied by IST methods?